The 4-Color Theorem: History and Implications

Can Computers Prove Theorems?

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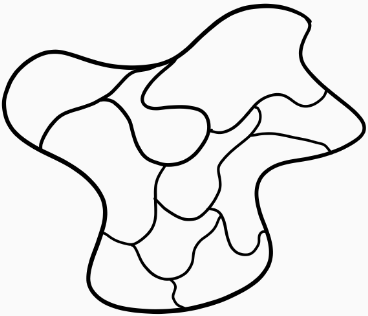
Outline

A History

Coloring Blobland



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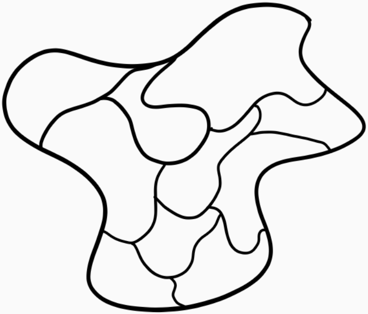
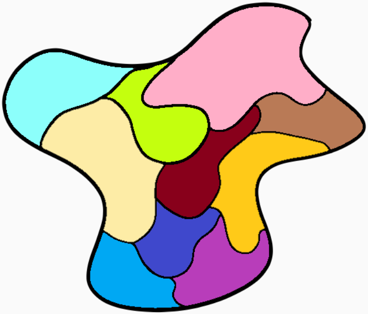
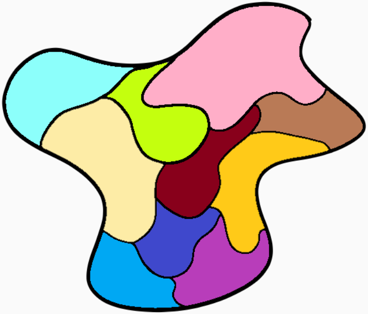


Goal: Color its states so that no two neighboring states get the same color.

Coloring Blobland

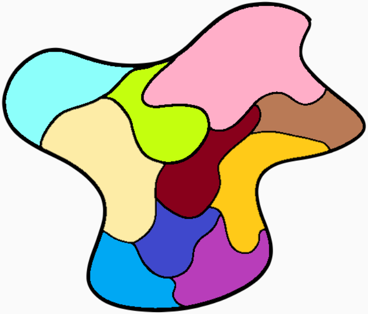


An obvious solution - color all the states with different colors.



Can we do better?

Coloring Blobland



Can we do better? Yes!

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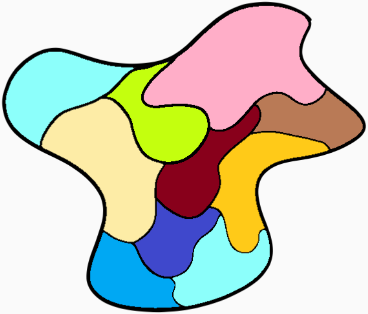

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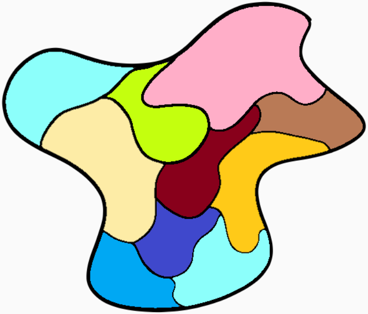

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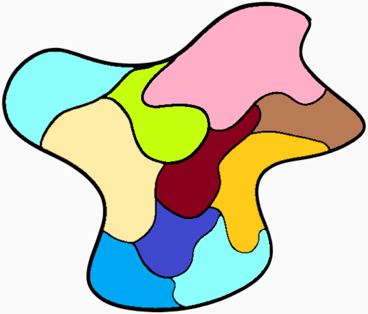

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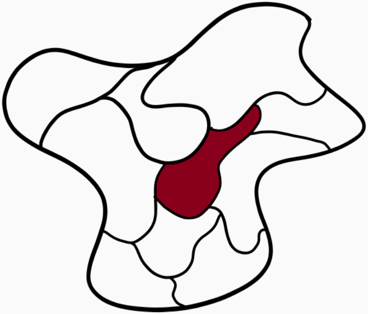
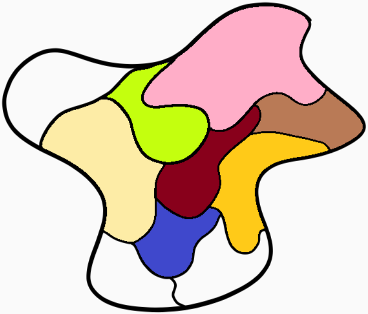


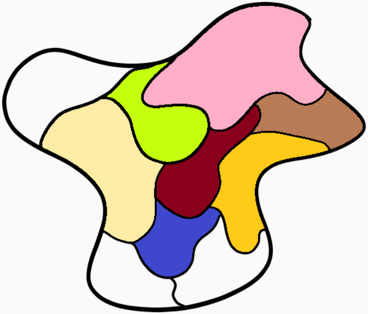
Consider the state with the most number of neighbors.

Coloring Blobland



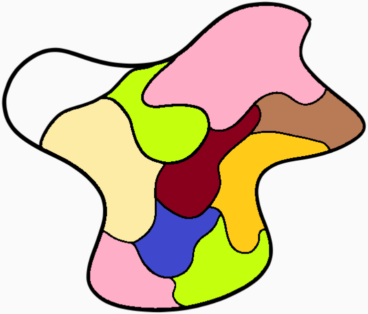
Color all of its neighbors with unique colors.

Coloring Blobland



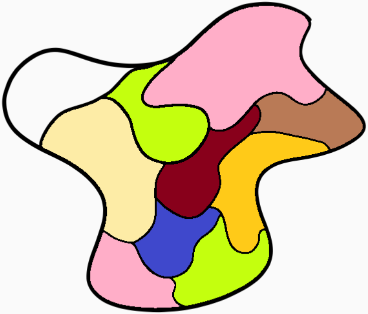
Two of the five neighbors of the dark-blue state are uncolored.

Coloring Blobland



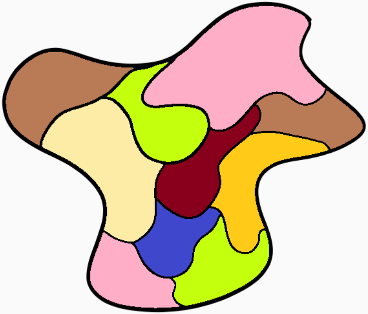
Use two of the unused colors to color them!

Coloring Blobland



The light-yellow state has one uncolored neighbor...

Coloring Blobland



Use one of the unused colors to color it!

Coloring A Map

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- Can we do better?

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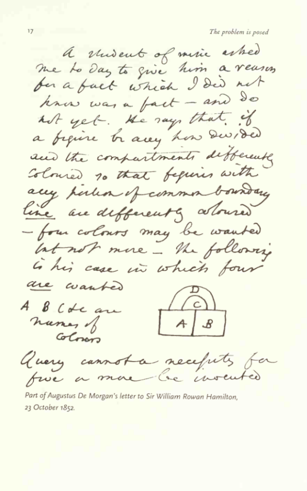
- De Morgan shared the problem to William Hamilton.


17 The problem is posed

A student of mine asked me to day to give him a reason for a fact which I did not know was a fact — and do not yet. He says that if a figure be any how divided and the compartments differently coloured so that figures with any portion of common boundary line are differently coloured — four colours may be wanted but not more — the following is his case in which four are wanted

A B C D are names of colours

Query cannot a necessity for five or more be invented

Part of Augustus De Morgan's letter to Sir William Rowan Hamilton, 23 October 1852.

The Search Continues...

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- In 1878, Arthur Cayley revived the search for the proof of this conjecture.
- His student Alfred Kempe published a proof of the conjecture in *Nature* the following year.

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- In 1890 Percy Heawood proved that Kempe's proof was incorrect.
- He salvaged enough of it and proved that **five colors** are sufficient to color any map.
- We will prove that six colors suffice for any map coloring and sketch the proof of Heawood's five-color theorem.

Questions?

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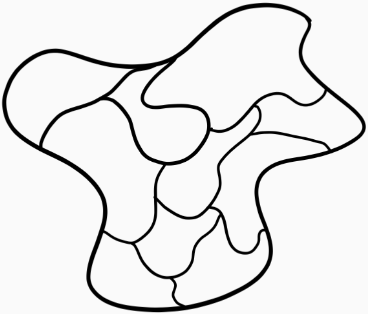
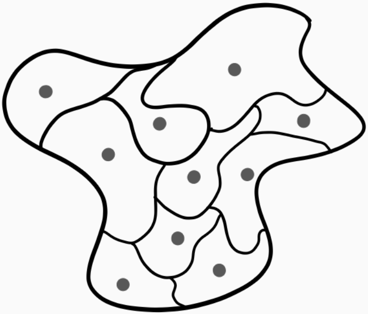
Planar Graphs



Maps to Graphs



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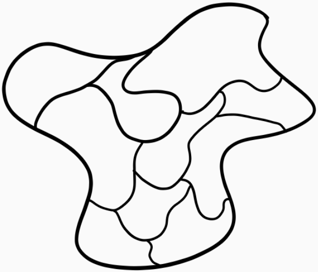

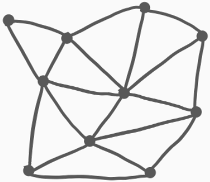

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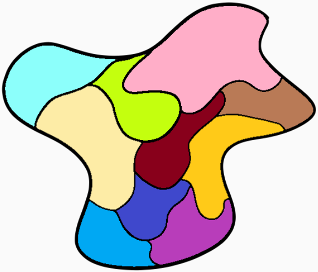

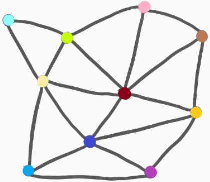

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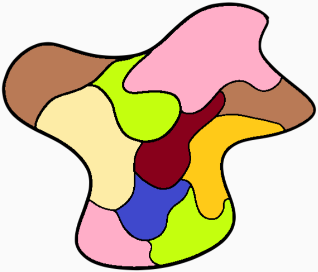

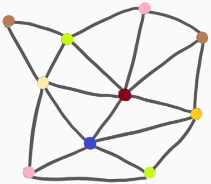

Maps to Graphs

Graphs which can be constructed from maps are called *planar*.

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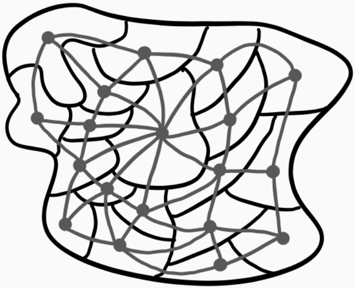

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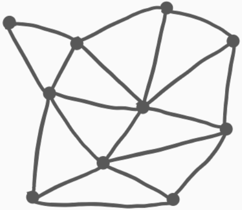

Equivalently, graphs were the vertices are drawn on the plane and the edges do not cross are planar.

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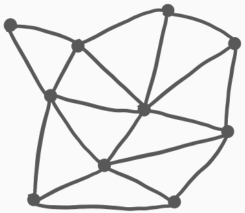

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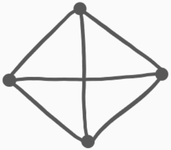

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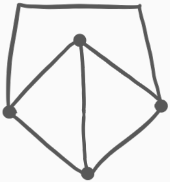

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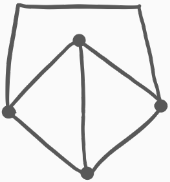

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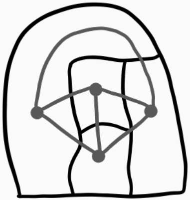

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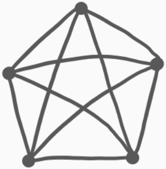
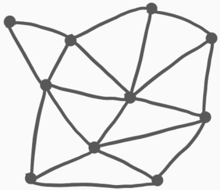
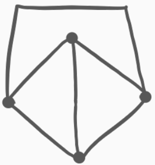


Not all graphs are planar!

Planar Graphs





Planar graphs satisfy *Euler's Formula*.

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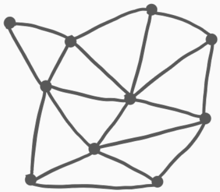

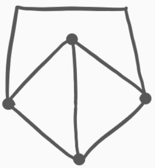

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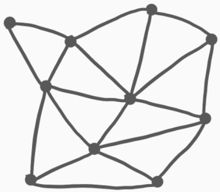

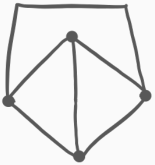

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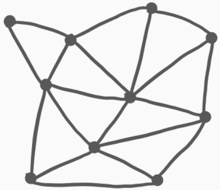
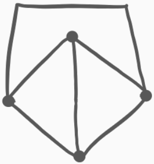
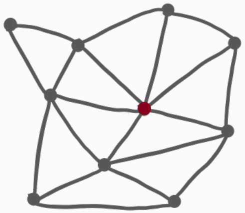




$|V(G_1)| = 10$ and $|E(G_1)| = 19$; $|V(G_2)| = 4$ and $|E(G_2)| = 6$.

Some Definitions



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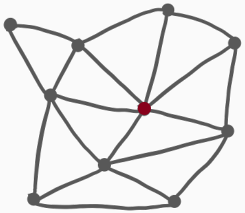

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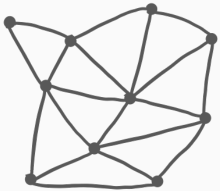

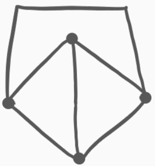

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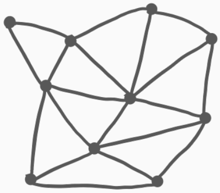

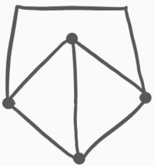

Let $\bar{\Delta}(G) = \frac{\sum_{v \in V(G)} d(v)}{|V(G)|}$, the average degree of the graph. Here, $\bar{\Delta}(G_1) = \frac{2 \times 19}{10}$ and $\bar{\Delta}(G_2) = \frac{2 \times 6}{4}$.

An Observation

Observation

Let G be a graph. Then, $2|E(G)| = \sum_{v \in V(G)} d(v)$. Therefore,

$$\bar{\Delta}(G) = \frac{2|E(G)|}{|V(G)|}$$

A Vertex of Small Degree

Let G be a planar graph. Then there exists $v \in V(G)$ such that $d(v) \leq 5$.

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Since the average degree is strictly less than 6, there exists a vertex v with $d(v) \leq 5$.

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Any map can be colored with at most six colors.

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- G' can be colored with six colors; hence, G with six colors.

Questions?

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Key Ideas

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Back to Maps

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Unavoidable Sets

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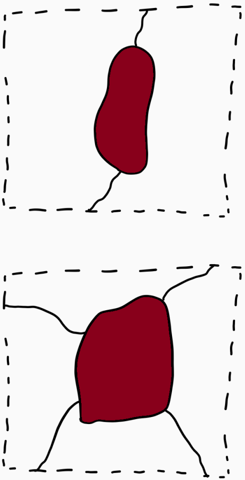
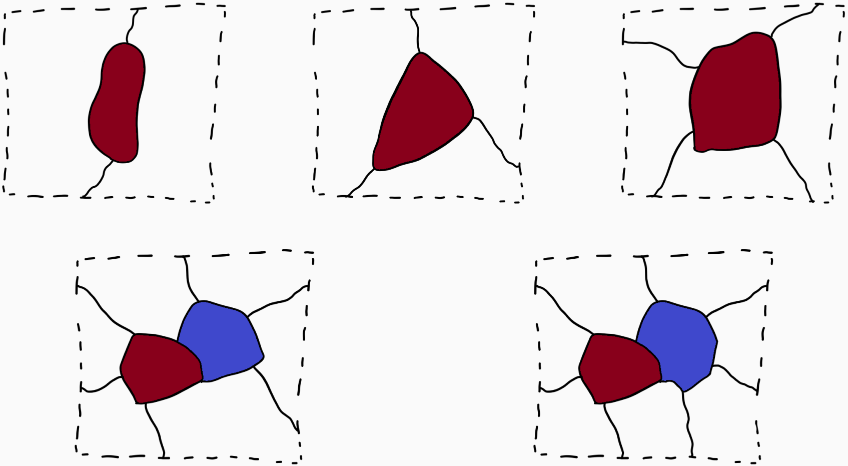
- Thus, every map must contain at least one of a "monogon", "digon", "triangle", "square", or "pentagon".
- This set is called an *unavoidable set*.
- We care about them since we will encounter them in every map!

Kempe's Unavoidable Set



Paul Wernicke's Unavoidable Set



More Unavoidable Sets

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More Unavoidable Sets

- In 1920, Philip Franklin produced an unavoidable set with nine configurations.
- In 1940, Henri Lebesgue constructed several interesting unavoidable sets.
- By the 1960s, unavoidable sets with thousands of configurations were produced.

Minimal Criminals

- Assume that the four-color theorem is false.

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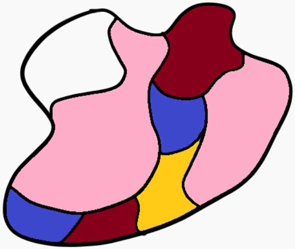
Minimal Criminals

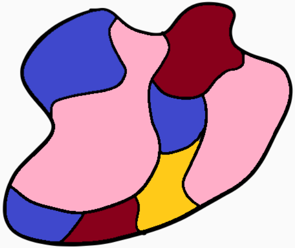
- Assume that the four-color theorem is false.
- There exists a map which requires at least five colors to color.
- Such a map with the least number of states (say k) is called a *minimal criminal* of the problem.
- **Every map** with at most $k-1$-many vertices is four-colorable!

Monogon?



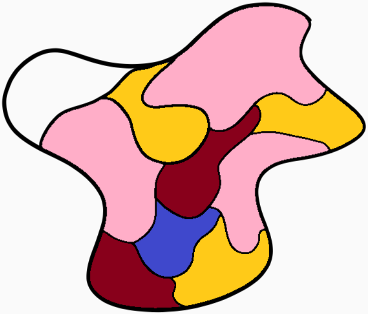
Can a monogon appear in a minimal criminal?

Monogon?



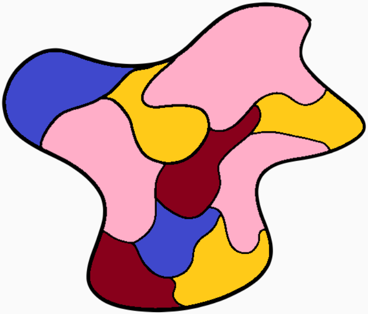
Can a monogon appear in a minimal criminal? No!

Digon?



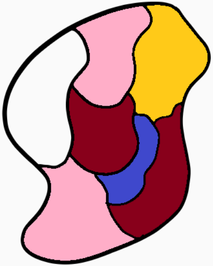
Can a digon appear in a minimal criminal?

Digon?



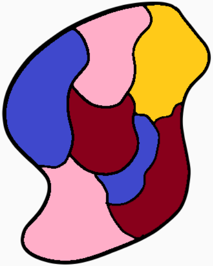
Can a digon appear in a minimal criminal? No!

Triangle?



Can a triangle appear in a minimal criminal?

Triangle?



Can a triangle appear in a minimal criminal? No!

Reducible Configurations

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- Kempe also proved that squares cannot appear in a minimal criminal.
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- Are there more reducible configurations? Yes!

Reducible Configurations

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- In 1920, Philip Franklin used his ideas to prove that all maps with at most 24 states is four-colorable.
- In 1938, he increased this to 35 states.

Unavoidable Set

A set of configurations is unavoidable if every map contains at least one configuration from this set.

Reducible Configuration

A configuration is called reducible if it cannot appear in a minimal criminal.

Merging Concepts

Unavoidable Set

A set of configurations is unavoidable if every map contains at least one configuration from this set.

Reducible Configuration

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- What if we can construct an unavoidable set of reducible configurations?

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A set of configurations is unavoidable if every map contains at least one configuration from this set.

Reducible Configuration

A configuration is called reducible if it cannot appear in a minimal criminal.

- What if we can construct an unavoidable set of reducible configurations?
- Then every map must contain a reducible configuration...
- Thus, the 4-Color Theorem will be proved!

Questions?

Questions?

Four Colors Suffice

The Search Begins

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- Wolfgang Haken was an attendee of this lecture.
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- He reached out to Heesch and invited him to Illinois.

The Computers Arrive

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- With the help of Karl Dürre and a CDC 1604A, he started checking large configurations for reducibility.
- Worked on a CDC 6600 with Yoshio Shimamoto the following two years.
- In late 1971, Shimamoto proved that if a particular configuration were reducible, then the four-color problem was solved!

The Computers Take Over

- Karl's program showed that the configuration was not reducible, halting progress again.

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A Quote

I don't know of anything involving computers that can't be done; some things just take longer than others. Why don't we take a shot at it?

The Computers Take Over

- Appel and Haken were able to prove that there exists an unavoidable set which uses only “good” configurations in 1974.

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The Computers Take Over

- Appel and Haken were able to prove that there exists an unavoidable set which uses only “good” configurations in 1974.
- To help with checking reducibility, they roped in John Koch, a graduate student.
- By 1976, they had used 487 rules to construct the unavoidable set.
- With the help of Haken’s daughter Dorothea, they checked, by hand, the 2000 odd configurations for reducibility.

Success!

A Quote

Modulo careful checking, it appears that four colors suffice!

Success!

- After a month rewriting the pre-print, they announced their result on June 21, 1976.

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Success!

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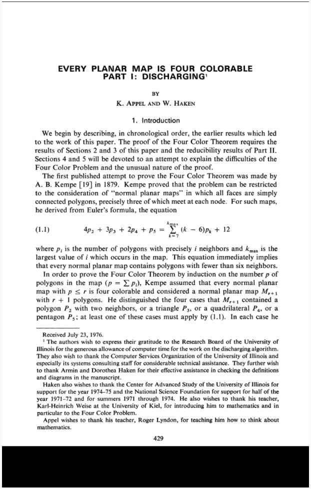
- A long, arduous peer-review process later...
- The final version of the paper was published in December, 1977.

Success!



EVERY PLANAR MAP IS FOUR COLORABLE PART I: DISCHARGING[1]

BY

K. APPEL AND W. HAKEN

1. Introduction

We begin by describing, in chronological order, the earlier results which led to the work of this paper. The proof of the Four Color Theorem requires the results of Sections 2 and 3 of this paper and the reducibility results of Part II. Sections 4 and 5 will be devoted to an attempt to explain the difficulties of the Four Color Problem and the unusual nature of the proof.

The first published attempt to prove the Four Color Theorem was made by A. B. Kempe [19] in 1879. Kempe proved that the problem can be restricted to the consideration of "normal planar maps" in which all faces are simply connected polygons, precisely three of which meet at each node. For such maps, he derived from Euler's formula, the equation

$$4p_2 + 3p_3 + 2p_4 + p_5 = \sum_{k=7}^{k_{max}} (k-6)p_k + 12 \qquad (1.1)$$

where p_i is the number of polygons with precisely i neighbors and k_{max} is the largest value of i which occurs in the map. This equation immediately implies that every normal planar map contains polygons with fewer than six neighbors.

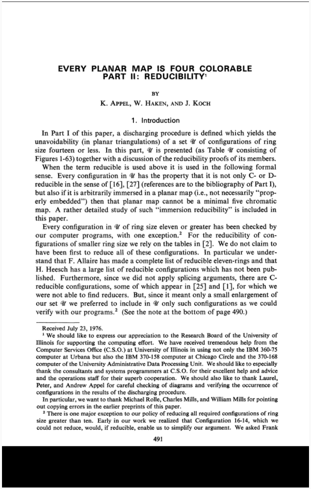
In order to prove the Four Color Theorem by induction on the number p of polygons in the map ($p = \sum p_i$), Kempe assumed that every normal planar map with $p \leq r$ is four colorable and considered a normal planar map M_{r+1} with $r+1$ polygons. He distinguished the four cases that M_{r+1} contained a polygon P_2 with two neighbors, or a triangle P_3, or a quadrilateral P_4, or a pentagon P_5; at least one of these cases must apply by (1.1). In each case he

Received July 23, 1976.

[1] The authors wish to express their gratitude to the Research Board of the University of Illinois for the generous allowance of computer time for the work on the discharging algorithm. They also wish to thank the Computer Services Organization of the University of Illinois and especially its systems consulting staff for considerable technical assistance. They further wish to thank Armin and Dorothea Haken for their effective assistance in checking the definitions and diagrams in the manuscript.

Haken also wishes to thank the Center for Advanced Study of the University of Illinois for support for the year 1974–75 and the National Science Foundation for support for half of the year 1971–72 and for summers 1971 through 1974. He also wishes to thank his teacher, Karl-Heinrich Weise at the University of Kiel, for introducing him to mathematics and in particular to the Four Color Problem.

Appel wishes to thank his teacher, Roger Lyndon, for teaching him how to think about mathematics.



EVERY PLANAR MAP IS FOUR COLORABLE PART II: REDUCIBILITY[1]

BY

K. APPEL, W. HAKEN, AND J. KOCH

1. Introduction

In Part I of this paper, a discharging procedure is defined which yields the unavoidability (in planar triangulations) of a set $\mathscr{U}$ of configurations of ring size fourteen or less. In this part, $\mathscr{U}$ is presented (as Table $\mathscr{U}$ consisting of Figures 1-63) together with a discussion of the reducibility proofs of its members.

When the term reducible is used above it is used in the following formal sense. Every configuration in $\mathscr{U}$ has the property that it is not only C- or D-reducible in the sense of [16], [27] (references are to the bibliography of Part I), but also if it is arbitrarily immersed in a planar map (i.e., not necessarily "properly embedded") then that planar map cannot be a minimal five chromatic map. A rather detailed study of such "immersion reducibility" is included in this paper.

Every configuration in $\mathscr{U}$ of ring size eleven or greater has been checked by our computer programs, with one exception.[2] For the reducibility of configurations of smaller ring size we rely on the tables in [2]. We do not claim to have been first to reduce all of these configurations. In particular we understand that F. Allaire has made a complete list of reducible eleven-rings and that H. Heesch has a large list of reducible configurations which has not been published. Furthermore, since we did not apply splicing arguments, there are C-reducible configurations, some of which appear in [25] and [1], for which we were not able to find reducers. But, since it meant only a small enlargement of our set $\mathscr{U}$ we preferred to include in $\mathscr{U}$ only such configurations as we could verify with our programs.[2] (See the note at the bottom of page 490.)

Received July 23, 1976.

[1] We should like to express our appreciation to the Research Board of the University of Illinois for supporting the computing effort. We have received tremendous help from the Computer Services Office (C.S.O.) at University of Illinois in using not only the IBM 360-75 computer at Urbana but also the IBM 370-158 computer at Chicago Circle and the 370-168 computer of the University Administrative Data Processing Unit. We should like to especially thank the consultants and systems programmers at C.S.O. for their excellent help and advice and the operations staff for their superb cooperation. We should also like to thank Laurel, Peter, and Andrew Appel for careful checking of diagrams and verifying the occurrence of configurations in the results of the discharging procedure.

In particular, we want to thank Michael Rolle, Charles Mills, and William Mills for pointing out copying errors in the earlier preprints of this paper.

[2] There is one major exception to our policy of reducing all required configurations of ring size greater than ten. Early in our work we realized that Configuration 16-14, which we could not reduce, would, if reducible, enable us to simplify our argument. We asked Frank

Is it a Proof?

- The response was, at best, muted.

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- There were two groups: those who did not believe that the thousands of cases solved by the computer was error-free...

Is it a Proof?

- The response was, at best, muted.
- There were two groups: those who did not believe that the thousands of cases solved by the computer was error-free...
- And those who were unconvinced that the 700 pages of hand calculations was error-free!

What is a Proof Today?

A Quote

...it seems that the computer-assisted work of Appel, Haken and Koch on the well-known Four-Color Problem may represent a watershed in the history of mathematics. Their work has been remarkably successful in forcing us to ask: What is a Proof Today?

Thomas Tymoczko's Paper

THE JOURNAL OF PHILOSOPHY

VOLUME LXXVI, NO. 2, FEBRUARY 1979

THE FOUR-COLOR PROBLEM AND ITS PHILOSOPHICAL SIGNIFICANCE *

THE old four-color problem was a problem of mathematics for over a century. Mathematicians appear to have solved it to their satisfaction, but their solution raises a problem for philosophy which we might call the *new four-color problem*.

The old four-color problem was whether every map on the plane or sphere can be colored with no more than four colors in such a way that neighboring regions are never colored alike. This problem is so simple to state that even a child can understand it. Nevertheless, the four-color problem resisted attempts by mathematicians for more than one hundred years. From very early on it was proved that five colors suffice to color a map, but no map was ever found that required more than four colors. In fact some mathematicians thought that four colors were not sufficient and were working on methods to produce a counterexample when Kenneth Appel and Wolfgang Haken, assisted by John Koch, published a proof that four colors suffice.† Their proof has been accepted by most mathematicians, and the old four-color problem has given way in mathematics to the new four-color theorem (4CT).

The purpose of these remarks is to raise the question of whether the 4CT is really a theorem. This investigation should be purely philosophical, since the mathematical question can be regarded as definitively solved. It is not my aim to interfere with the rights of

* I would like to thank Michael Albertson, Joan Hutchinson, and William Marsh for reading a draft of this paper and for some helpful discussions on a number of points.

† "Every Planar Map Is Four Colorable," *Illinois Journal of Mathematics*, XXI, 84 (September 1977): 429–567. Part I, on Discharging, is by Appel and Haken; part II, on Reducibility, was done in conjunction with Koch. Parenthetical page references to Appel, Haken, and Koch, will be to this article.

0022-362X/79/7602/0057$02.60

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What is a Proof?

A valid proof must be *convincing* and *surveyable*.

Concerns

- No qualms about the construction of the unavoidable set.

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- Would mathematics become an empirical science?

Concerns

- No qualms about the construction of the unavoidable set.
- Would mathematics become an empirical science?
- Can a proof be considered valid if it cannot be checked by hand?

Quotes

Ted Swart

For the most part I regard computer-assisted proof as just an extension of pencil and paper. I don't think there's some great divide which says that OK, you're allowed to use pencil and paper but you're not allowed to use a computer because that changes the character of the proof. I don't see that myself. I find such an argument strange.

Quotes

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Ted Swart

Human beings get tired, and their attention wanders, and they are all too prone to slips of various kinds... Computers do not get tired.

Quotes

Ian Stewart

The answer appears as a kind of monstrous coincidence. Why is there an unavoidable set of reducible configurations? The best answer at the present time is: there just is. The proof: here it is, see for yourself. The mathematician's search for hidden structure, his pattern-binding urge, is frustrated.

Quotes

Ian Stewart

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Daniel Cohen

... the real thrill of mathematics is to show that as a feat of pure reasoning it can be understood why four colors suffice. Admitting the computer shenanigans of Appel and Haken to the ranks of mathematics would only leave us intellectually unfulfilled.

Kenneth Appel

… there were people who said, “This is terrible mathematics, because mathematics should be clean and elegant”, and I would agree. It would be nicer to have clean and elegant proofs.

Improvements

- In 1989, Haken and Appel corrected all errors and published their last word on the subject through a book.

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Improvements

- In 1989, Haken and Appel corrected all errors and published their last word on the subject through a book.
- In 1994, Robertson *et al.* published a proof of the theorem using a smaller unavoidable set using a a tenth of the rules. Their algorithm also ran much quicker.
- In 2004, Georges Gonthier used a “proof checker” to verify that the proof of the four-color theorem was valid!

Food For Thought

- Why should proofs be elegant?

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- Why should proofs be elegant?
- What differentiates arduous hand-written proofs and computer-aided proofs?
- Assume that “maps” are now three-dimensional. Can all such maps be colored using 4-colors?
- What if maps are embedded in different spaces? A torus? Other surfaces with higher genus?

Questions?

Please feel free to send me a mail[1] if you have any questions regarding this talk or just want to discuss the topic!

Thank you for your time!

Thanks to Chi-Ning for the opportunity!

[1]The ID is kprahlad.narasimhan@niser.ac.in just in case the link is broken.

References

For the general audience:

- Blog by Jesus Najera. Basics and history.
- Slides by Robin Wilson. Basics and history.
- Book by Robin Wilson. A comprehensive account of the history of the four-color theorem.
- From the Horse's Mouth. A Scientific American article by Appel and Haken.

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- Lecture Notes by Moti Ben-Ari. Some basics and the proof of the five-color theorem.
- The Final Word. Every Planar Map is Four Colorable, Kenneth Appell and Wolfgang Haken, Contemporary Mathematics, 1989.
- An Improved Proof. A New Proof Of The Four-Colour Theorem, Robertson *et al.*, Announcements of the American Mathematical Society, 1996.

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- A Final Check. A computer-checked proof of the Four Colour Theorem by Georges Gonthier.
- The Holy Graph Theory Book. Graph Theory, Reinhard Diestel, Graduate Texts in Mathematics, 2000. Chapter 4 contains details on planar graphs. Chapter 5 contains details on coloring.

References

For those interested in the philosophical implications:

- What is a Proof? The Four-Color Problem and Its Philosophical Significance, Thomas Tymoczko, The Journal of Philosophy, 1979.
- A Rebuttal. Swart's response to Tymoczko's paper in support of the proof.
- Are Proofs Dying? Mathematicians and computer scientists weigh on whether computers will replace mathematicians.