A FPT Algorithm for the Orthogonal Terrain Guarding Problem

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In this report, we start by familiarising ourselves with a few definitions related to terrain guarding. We will then move onto proving some basic lemmas in this field. We will conclude the report by describing an exponential FPT algorithm to solve the DISCRETE TERRAIN GUARDING problem for orthogonal terrains.

1 Introduction

Definition. Let $V = \{v_1, \ldots, v_n\}$ be a finite sequence of points in $\mathbb{R}^2$. The curve specified by the line segments connecting v_i and v_{i+1} for all $1 \leq i < n$ is called a *polygonal chain*. This polygonal chain is said to be defined by V.

We define $x(p)$ and $y(p)$ to be the x and y coordinates of a point p in $\mathbb{R}^2$.

Definition. A *1.5D terrain* T is a polygonal chain defined by $V = \{v_1, \ldots, v_n\}$, a finite sequence of three or more points in $\mathbb{R}^2$ which have the following two properties: $x(v_i) \leq x(v_j)$ for all $1 \leq i < j \leq n$ and $x(v_{i-1}) = x(v_i) = x(v_{i+1})$ for some $1 < i < n$ implies that $y(v_{i-1}) < y(v_i) < y(v_{i+1})$ or $y(v_{i-1}) > y(v_i) > y(v_{i+1})$.

Throughout the report, a terrain will refer to a 1.5D terrain. We use $\operatorname{IM} T$ to denote the image of T. It is useful to think of T as a graph with vertices V and edges $E = \{(v_i, v_{i+1}) \mid 1 \leq i < n\}$. For $a, b \in \operatorname{IM} T$, we say a precedes b, denoted by $a \prec b$, if a appears on the terrain before b does (the terrain is scanned from left to right). Two points on the terrain see each other if the line segment joining these points lies above the terrain. This is defined formally below.

Definition. Let $a, b \in \operatorname{IM} T$ where $a \prec b$. a *sees* b if $\overline{ab}(x(t)) \geq y(t)$ for all t such that $a \prec t \prec b$. $\overline{ab}$ denotes the line joining a and b.

Clearly, the relation"seeing" is reflexive and symmetric. It is, unfortunately, not transitive as illustrated by Figure 1.

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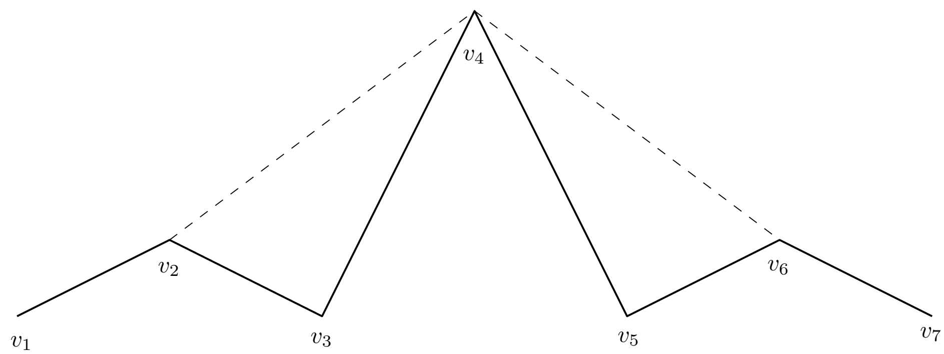

Figure 1: v_2 sees v_4 and v_4 sees v_6 but v_2 does not see v_6.

Definition. Let $U \subseteq \operatorname{Im} T$. Then the *visibility region* of U, $\operatorname{vis} U = \{v \in \operatorname{Im} T \mid \text{there exists } u \in U \text{ seeing } v\}$.

If $U = \{u\}$, we abuse the notation and use $\operatorname{vis} u$ instead of $\operatorname{vis} U$.

Definition. A terrain $T(V, E)$ is defined to be *orthogonal* or *rectilinear* if for all $1 < i < n$, $x(v_{i-1}) = x(v_i)$ and $y(v_i) = y(v_{i+1})$ or $x(v_i) = x(v_{i+1})$ and $y(v_{i-1}) = y(v_i)$.

Equivalently, orthogonal terrains are those whose edges are either parallel to the x or y axis and an edge parallel to the x axis is followed by one parallel to the y axis and vice versa.

Definition. In an orthogonal terrain $T(V, E)$, a vertex v_i $(1 < i < n)$ is:

(i) *Convex* if the angle formed by v_{i-1}, v_i and v_{i+1} above the terrain is convex.

(ii) *Reflex* if the angle formed by v_{i-1}, v_i and v_{i+1} above the terrain is reflex.

(iii) *Left* if $x(v_{i-1}) = x(v_i)$.

(iv) *Right* if $x(v_i) = x(v_{i+1})$.

The set of reflex vertices is denoted by $\mathscr{R}$ and the set of convex vertices is denoted by $\mathscr{C}$. The set of vertices which are both reflex and left are called left reflex vertices and are denoted by $\mathscr{R}_l$. Right reflex vertices are symmetrically defined and are denoted by $\mathscr{R}_r$. The set of left convex and right convex vertices are similarly defined and are denoted by $\mathscr{C}_l$ and $\mathscr{C}_r$ respectively. This is illustrated in Figure 2. A left convex vertex is defined to be of the opposite type as a right reflex vertex while a right convex vertex is the opposite type as a left reflex vertex. v_1 and v_n are defined to be the opposite type as v_2 and v_{n-1} respectively (in Figure 2, $v_1 \in \mathscr{C}_l$ and $v_8 \in \mathscr{C}_r$). We now define a couple of different versions of the terrain guarding problem. These are referenced from [1].

Problem. Continuous Terrain Guarding: Given a terrain $T(V, E)$ with $|V| = n$ and $k \in \mathbb{N}$, decide if there exists a subset $S \subseteq \operatorname{Im} T$ with $|S| \leq k$ such that $\operatorname{vis} S \supseteq V$.

Problem. Discrete Terrain Guarding: Given a terrain $T(V, E)$ with $|V| = n$ and $k \in \mathbb{N}$, decide if there exists a subset $S \subseteq V$ with $|S| \leq k$ such that $\operatorname{vis} S \supseteq V$.

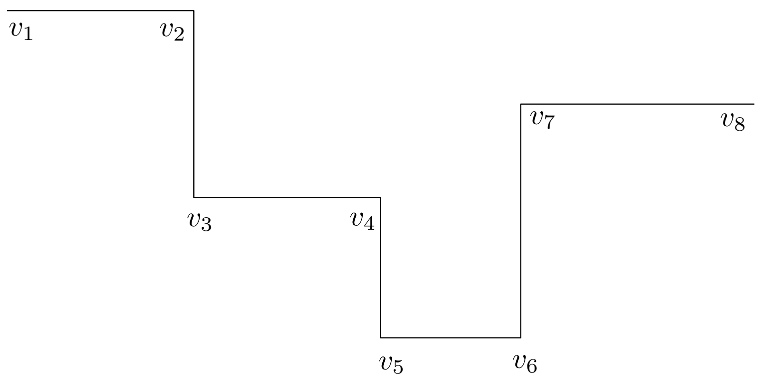

Figure 2: Here, $v_7 \in \mathscr{R}_l$; $v_2, v_4 \in \mathscr{R}_r$; $v_1, v_3, v_5 \in \mathscr{C}_l$; $v_6, v_8 \in \mathscr{C}_r$.

Here, it is important to note the difference between the two versions of the problem: in Continuous Terrain Guarding, we can place guards (that is, choose S from) anywhere on the terrain to guard V while in Discrete Terrain Guarding, we are allowed to place guards only on the vertices of the terrain and are required to guard V. Clearly, the continuous version of the problem is more general than the latter. A Continuous Terrain Guarding instance is denoted by $\overline{(T(V,E), n, k)}$ while a Discrete Terrain Guarding instance is denoted by $(T(V,E), n, k)$.

Problem. Annotated Terrain Guarding: Given a terrain $T(V,E)$ with $|V| = n$ and $k \in \mathbb{N}$, $\mathcal{G}, \mathcal{C} \subseteq V$ decide if there exists a $S \subseteq \mathcal{G}$ with $|S| \leq k$ such that $\textsc{vis}\, S \supseteq \mathcal{C}$.

In Annotated Terrain Guarding, we are further restricting the placements of the guards to $\mathcal{G}$, a subset of V, and require it to guard only $\mathcal{C}$ instead of the whole set V. An Annotated Terrain Guarding instance is denoted by $(T(V,E), n, k, \mathcal{G}, \mathcal{C})$. The Annotated Terrain Guarding instance where $\mathcal{G} = \mathcal{C} = V$ is equivalent to the Discrete Terrain Guarding instance $(T(V,E), n, k)$. We will now define a useful graph corresponding to a terrain with respect to the Annotated Terrain Guarding problem.

Definition. Let $(T(V,E), n, k, \mathcal{G}, \mathcal{C})$ be an Annotated Terrain Guarding instance. The *visibility graph* of the instance, G_T, is defined as follows: $G_T = (\mathcal{C}, E')$ where $E' = \{(u,v) \in \mathcal{C}^2 \mid \text{there is a } g \in \mathcal{G} \text{ seeing } u \text{ and } v\}$.

The visibility graphs of the terrains in Figures 1 and 2 are illustrated in Figure 3.

Algorithm 1: Visibility Graph of a Terrain

Input : An Annotated Terrain Guarding instance $(T(V,E), n, k, \mathcal{G}, \mathcal{C})$
Output: Visibility graph of this instance.

```
E'[ ]
for each distinct pair of vertices (v_i, v_j) ∈ C^2, i < j do
    if there is a g ∈ G seeing v_i and v_j then
        ADD (v_i, v_j) to E'[ ]
    end if
end for
return G_T(C, E')
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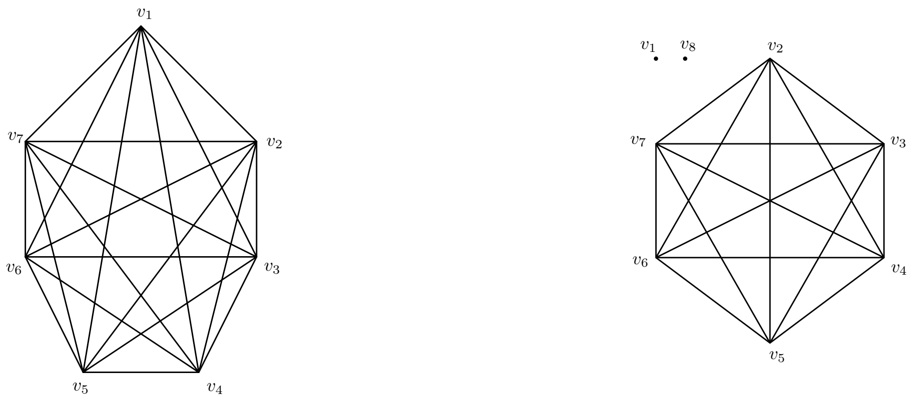


Figure 3: G_T for the terrains in Figures 1 and 2 where $\mathcal{G} = \{v_2, v_3, v_4, v_5\}$ and $\mathcal{C} = V$.

Algorithm 1 computes the visibility graph of an Annotated Terrain Guarding instance. Since checking whether two vertices in a terrain see each other takes $\mathcal{O}(|V|)$ time, the algorithm runs in $\mathcal{O}(|\mathcal{C}|^2|\mathcal{G}||V|)$ time. Since $\mathcal{C}$ and $\mathcal{G}$ are subsets of V, this runs in $\mathcal{O}(|V|^4)$ time.

If $G(V, E)$ is a graph and $V' \subseteq V$, we denote the graph induced by V' as $G[V']$. Finally, we describe the clique cover problem and define chordal graphs.

Problem. Clique Cover: Given a graph $G(V, E)$ with $|V| = n$ and $k \leq n$, $k \in \mathbb{N}$, decide if there exists a collection of cliques $\mathcal{K} = \{K_1, K_2 \dots K_k\}$ in G such that for any $v \in V$, $v \in K_i$ for some $K_i \in \mathcal{K}$.

A Clique Cover instance is denoted by $(G(V, E), n, k)$. Since a graph G can be covered by k cliques if, and only if, the complement of G is $k-$colourable, this problem is NP-Hard.

Definition. A graph $G(V, E)$ is *chordal* if for all $V' \subseteq V$ where $|V'| \geq 4$, $G[V']$ is not a cycle.

Observation 1.1. Let $G(V, E)$ be a chordal graph. For any $V' \subseteq V$, $G[V']$ is chordal.

Note that, to check if a given graph $G(V, E)$ is chordal, it is enough to check if for any cycle $C \subseteq V$ where $|C| \geq 4$, $G[C]$ is a not a cycle.

2 Terrains and Chordal Graphs

In this section, we look at a few properties of terrains and chordal graphs. We then establish a relationship between these two concepts. We first state and prove the Order Claim, referenced from [5], which lays the foundation for the theorems that follow.

Lemma 2.1 (Order Claim). *Let $T(V,E)$ be a terrain. Let $a, b, c, d \in \mathrm{Im}\, T$ where $a \prec b \prec c \prec d$ such that a sees c and b sees d. Then, a sees d.*

Proof. Since a sees c, $\overline{ac}(x(t)) \geq y(t)$ for all $a \prec t \prec c$. In particular, $\overline{ac}(x(b)) \geq y(b) = \overline{bd}(x(b))$. Thus, we have $\overline{ac}(x(b)) \geq \overline{bd}(x(b))$. Similarly, since b sees d, $\overline{bd}(x(c)) \geq \overline{ac}(x(c))$. Thus, d lies above $\overline{ac}$ and a lies above $\overline{bd}$. This implies that $\overline{ad}$ lies above $\overline{ac}$ and $\overline{bd}$. Since $\overline{ac}$ and $\overline{bd}$ lie above the terrain, $\overline{ad}$ lies above the terrain proving that a sees d. $\square$

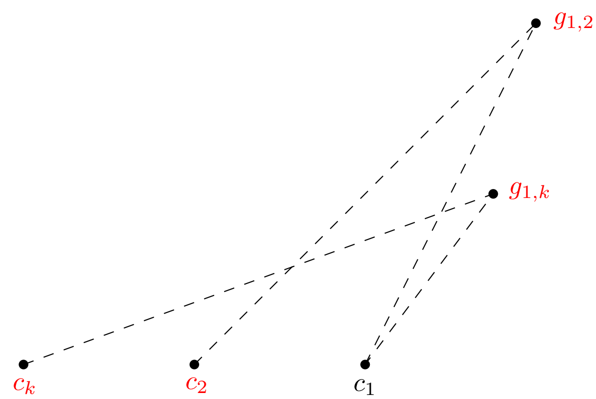


Figure 4

Definition. A *left guard* is a guard who can only see to its left, i.e, a left guard g sees $v \in \textsc{Im}\, T$ if $x(v) \leq x(g)$ and $\overline{vg}(x(t)) \geq y(t)$ for all $v \prec t \prec g$. A *right guard* is defined symmetrically.

Using the two theorems that follow, we will prove the equivalence between a restricted case of the Annotated Terrain Guarding problem and the clique cover problem in chordal graphs. We will prove in Observation 2.16 that we can solve a Clique Cover instance $(G(V, E), n, k)$ in $\mathcal{O}(|V|^2|E|)$ time if G is a chordal graph.

Theorem 2.2. *Let $(T(V, E), n, k, \mathcal{G}, \mathcal{C})$ be an* Annotated Terrain Guarding *instance where $\mathcal{G}$ is a set of left or right guards. Then, G_T is a chordal graph.*

Proof. We assume that $\mathcal{G}$ is a set of left guards. The proof for the other case follows using a symmetric argument. Let $C \subseteq \mathcal{C}$ where $|C| = k \geq 4$ be a cycle in G_T. We prove that $G_T[C]$ is not a cycle. Let $C = \{c_1, c_2 \dots c_k\}$ be the order of the vertices as they appear on the cycle. Also, we assume, without loss in generality, that $c_i \preceq c_1$ for all $c_i \in C$ and that $c_k \prec c_2$. By definition, there is a left guard $g_{1,k}$ which sees both c_1 and c_k. Similarly, we have $g_{1,2}$, a left guard, which sees both c_1 and c_2. Now, $c_1 \preceq g_{1,k}$ and $c_1 \preceq g_{1,2}$. If $g_{1,2} = g_{1,k}$ then we have an edge between c_2 and c_k in G_T. Since $|C| > 3$, $G_T[C]$ is not a cycle. We are now left with two cases:

Case 1. $g_{1,k} \prec g_{1,2}$.

As illustrated in Figure 4 (the vertices marked in red are the ones that we apply Lemma 2.1 on), $g_{1,2}$ sees c_k implying that there is an edge between c_2 and c_k in G_T. Since $k \geq 4$, $G_T[C]$ is not a cycle. Note that $g_{1,k}$ could be c_1.

Case 2. $g_{1,2} \prec g_{1,k}$.

In this case, we have two possibilities. The first one is that $c_1 \prec g_{1,2}$ while the second one is that $c_1 = g_{1,2}$. As shown in Figure 5a, if $c_1 \prec g_{1,2}$, then, once again, there is an edge between c_2 and c_k in $G_T[C]$. Now, consider the situation illustrated in Figure 5b. Unfortunately, we cannot use Lemma 2.1 directly. Thus, we look at c_3, the other neighbour of c_2. Note that c_3 exists as $|C| \geq 4$. Let $g_{2,3}$ be a left guard seeing both c_2 and c_3. We will show that if $c_3 \prec c_2$, then $G_T[C]$ is not a cycle. We will then use induction to show that for

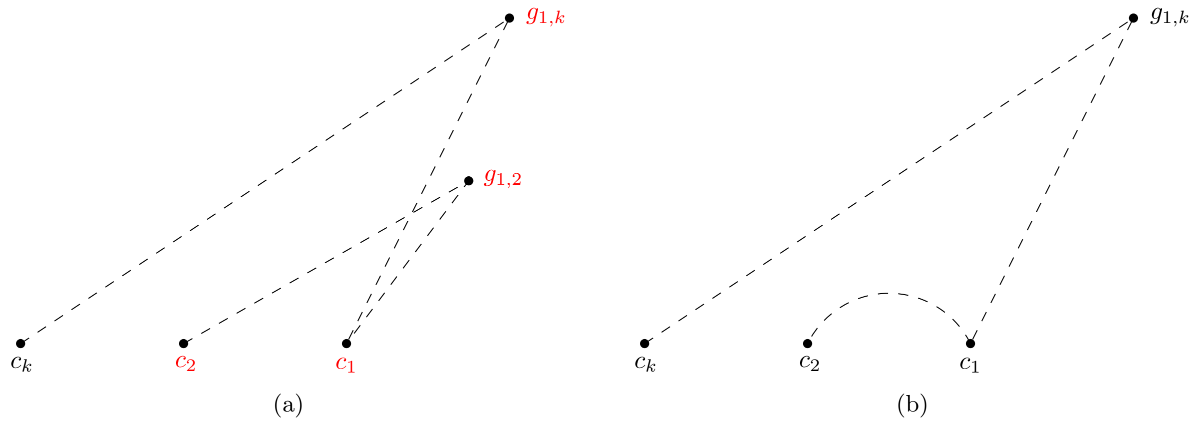


Figure 5

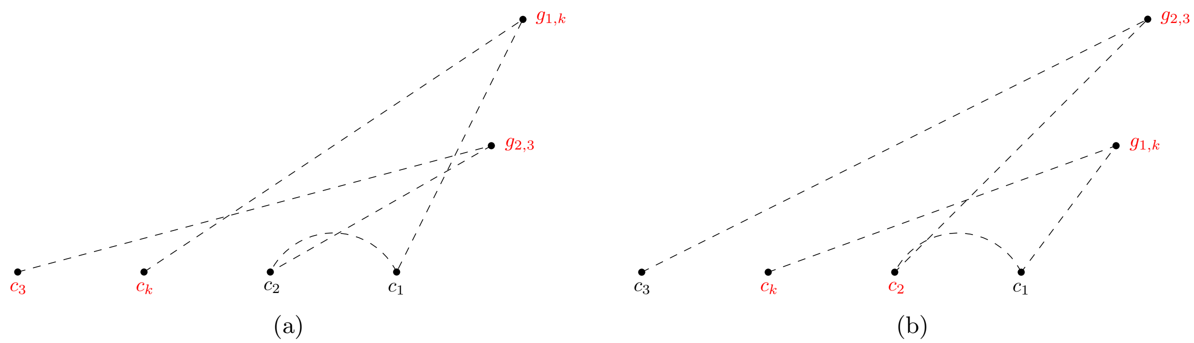


Figure 6

all c_i, where $3 \leq i \leq k$, the above claim, i.e, if $c_i \prec c_2$, then $G_T[C]$ is not a cycle, holds. Since $c_k \prec c_2$ by construction, this will complete our proof. We have the following two cases depending on the position of c_3:

Case 1. $c_3 \prec c_k$.

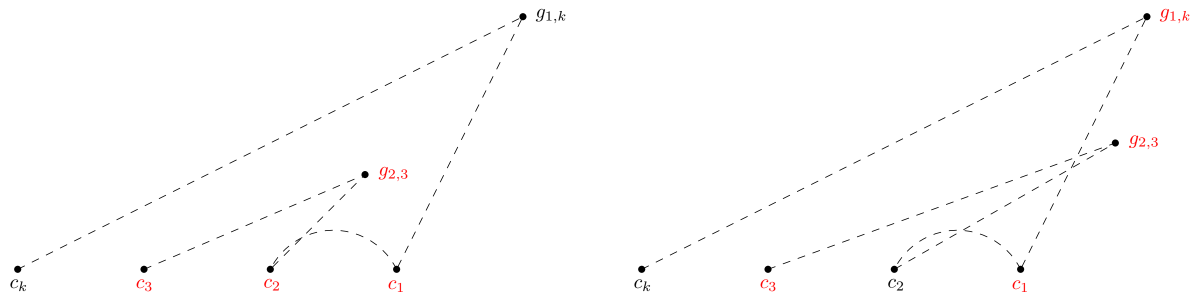
We have two cases: $g_{2,3} \prec g_{1,k}$ and $g_{1,k} \prec g_{2,3}$. These are shown in Figures 6a and 6b and using the Lemma 2.1, we get the existence of an edge between c_1 and c_3 in the former case and between c_2 and c_k in the latter case. Again, note that in the first case, $g_{2,3}$ could be c_2.

Case 2. $c_k \prec c_3 \prec c_2$.

We have three cases: $g_{2,3} \prec c_1$, $c_1 \prec g_{2,3} \prec g_{1,k}$, and $g_{1,k} \prec g_{2,3}$. Note that the third case is equivalent to the one in Figure 6b since the position of c_3 was not used in the proof of the existence of the (c_2, c_k) edge. The first two cases are shown in Figures 7a and 7b. Using Lemma 2.1, we get that there is an edge between c_1 and c_3 in both these cases.

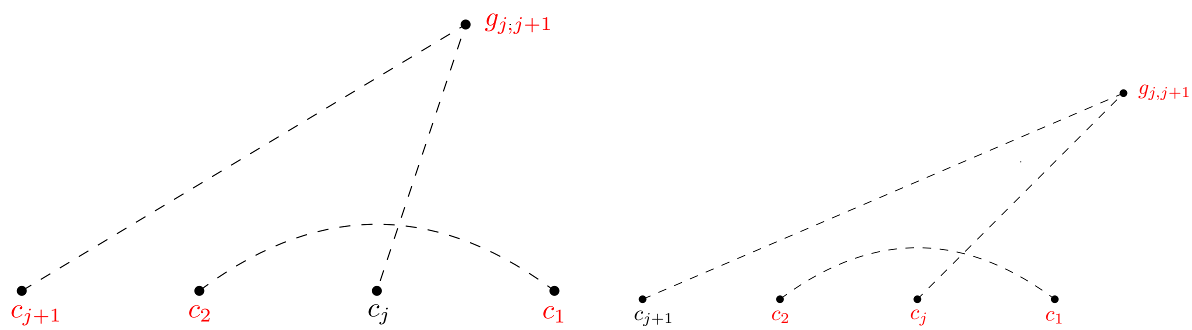
Thus, we have proven that if $c_3 \prec c_2$, then $G_T[C]$ is not a cycle. We now prove this claim for an arbitrary c_i, where $3 \leq i \leq k$ by induction on i.

We have proven the base case. Assume that, for some $3 \leq j \leq k-1$, our supposition is true. Thus, if $c_j \prec c_2$ then $G_T[C]$ is not a cycle. Now, assume that $c_2 \prec c_j$. Since we took c_1 to be the rightmost element of the



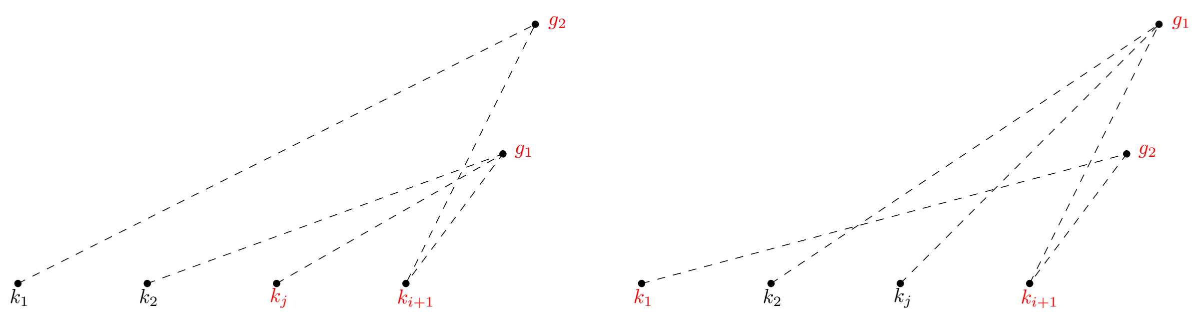
(a) (b)

Figure 7



(a) (b)

Figure 8



(a) (b)

Figure 9

cycle on the terrain, we have that $c_2 \prec c_j \prec c_1$. Assume that $c_{j+1} \prec c_2$. We have two cases as illustrated by Figures 8a and 8b. These prove that c_2 and c_{j+1} share an edge in $G_T[C]$. Since c_{j+1} is neither c_1 nor c_3, $G_T[C]$ is not a cycle. Note that $g_{j,j+1}$ could be c_j in Figure 8a. This proves our supposition.

Since $c_k \prec c_2$, $G_T[C]$ is not a cycle when C is a cycle of length at least 4. Thus, G_T is chordal. □

Theorem 2.3. *Let $(T(V,E), n, k, \mathcal{G}, \mathcal{C})$ be an* Annotated Terrain Guarding *instance where $\mathcal{G}$ is a set of left or right guards and $\mathcal{G} \cap \mathcal{C} = \emptyset$. Let $K \subseteq \mathcal{C}$ where $|K| \geq 2$. $G_T[K]$ is a clique iff there is a $g \in \mathcal{G}$ such that $\text{VIS}\, g \supseteq K$.*

Proof. Again, we prove this claim only for a set of left guards. The proof for the other case will follow symmetrically. Let K be a set such that there is a $g \in \mathcal{G}$ such that $\text{VIS}\, g \supseteq K$. Thus, for any pair of vertices in K there is an edge between them in $G_T[K]$ since there is a guard seeing them both. Thus, $G_T[K]$ is a clique.

Now, we prove the forward direction by induction on the number of vertices in K where $G_T[K]$ is a clique. If $|K| = 2$, then our claim follows trivially. Assume that our supposition holds for all cliques of size at most i where $i \geq 2$, $i \in \mathbb{N}$. Let $K = \{k_1, k_2, \ldots k_i, k_{i+1}\}$ where the vertices are ordered according to how they appear on the terrain. By the induction hypothesis, there is a left guard g_1 such that $\text{VIS}\, g_1 \supseteq \{k_2, \ldots k_i, k_{i+1}\}$. Since there is a (k_1, k_{i+1}) edge in $G_T[K]$, there is a left guard, say g_2, seeing both of them. If $g_1 = g_2$, then we have $\text{VIS}\, g_1 \supseteq K$ proving the supposition. We are now left with two cases:

Case 1. $g_1 \prec g_2$.

As illustrated in Figure 9a, using Lemma 2.1, we get that for any $2 \leq j \leq i$, g_2 sees k_j. Thus, $\text{VIS}\, g_2 \supseteq K$. Note that we have used that $g_1 \neq k_{i+1}$.

Case 2. $g_2 \prec g_1$.

Using Lemma 2.1 on the situation as in Figure 9b, we observe that g_1 sees k_1 implying that $\text{VIS}\, g_1 \supseteq K$. Note that we have used that $g_2 \neq k_{i+1}$.

This proves our supposition and completes the proof by induction. □

Theorems 2.2 and 2.3 are, to the best of my knowledge, an addition to existing literature. These two theorems give us the following observation.

Observation 2.4. Let $(T(V,E), n, k, \mathcal{G}, \mathcal{C})$ be an Annotated Terrain Guarding instance where $\mathcal{G}$ is a set of left or right guards and $\mathcal{G} \cap \mathcal{C} = \emptyset$. Then, $(T(V,E), n, k, \mathcal{G}, \mathcal{C}) \equiv (G_T(\mathcal{C}, E'), |\mathcal{C}|, k)$ where G_T is the visibility graph of T.

We will now look at a few properties of orthogonal terrains. We first observe that a left convex vertex can only see vertices to its right. Symmetrically, a right convex vertex can only see vertices to its left. Thus, to guard the set of left convex vertices, it is sufficient to have the guards to be left guards while to guard the set of right convex vertices, it is sufficient to have the guards to be right guards. Referring back to Figure 2 will make these observations straightforward.

Observation 2.5. Let $T(V, E)$ be an orthogonal terrain and $v \in \mathscr{C}_l$. Then, for all $w \in \textsc{vis}\, v$, $x(w) \geq x(v)$ and $y(w) \geq y(v)$. Symmetrically, if $v' \in \mathscr{C}_r$, for all $w' \in \textsc{vis}\, v'$, $x(w') \leq x(v')$ and $y(w') \geq y(v')$.

Observation 2.6. Let $T(V, E)$ be an orthogonal terrain and $\mathcal{G}$ be a set of guards such that $\textsc{vis}\, \mathcal{G} \supseteq \mathscr{C}_l$. Then, the set of left guards placed on $\mathcal{G}$ sees $\mathscr{C}_l$. Symmetrically, if $\mathcal{G}'$ is a set of guards such that $\textsc{vis}\, \mathcal{G}' \supseteq \mathscr{C}_r$, the set of right guards placed on $\mathcal{G}'$ sees $\mathscr{C}_r$.

Let $T(V, E)$ be an orthogonal terrain. We will now prove that the Annotated Terrain Guarding instance $(T(V, E), n, k, \mathscr{R}, \mathscr{C})$ is equivalent to both the Continuous Terrain Guarding and the Discrete Terrain Guarding versions of the problem.

Lemma 2.7. *Let $T(V, E)$ be an orthogonal terrain and $\mathcal{G} \subset \mathrm{Im}\, T$ be a finite set of guards $\textsc{vis}\, \mathcal{G} \supseteq \mathscr{C}$. Then, $\textsc{vis}\, \mathcal{G} \supseteq V$.*

Proof. Let $v_i \in \mathscr{R}_l$ where $i \neq 1$. Then, v_{i-1} is a right convex vertex. Symmetrically, if $v_i \in \mathscr{R}_r$ where $i \neq n$, v_{i+1} is a left convex vertex. Thus, apart from the two special cases, at least one of v_i's neighbours, say v_j, is convex and $x(v_i) = x(v_j)$ and $y(v_i) \geq y(v_j)$. Since $\textsc{vis}\, \mathcal{G} \supseteq \mathscr{C}$, there is a $g \in \mathcal{G}$ which sees v_j. Since v_i lies above $\overline{gv_j}$ and has the same x-coordinate, g sees v_i. Now, assume that $v_1 \in \mathscr{R}_l$. Then, by definition, $v_2 \in \mathscr{C}_r$. Thus, $\textsc{vis}\, v_2 = \{v_1, v_2, v_3\}$. Since $\textsc{vis}\, \mathcal{G} \supseteq \mathscr{C}$, $\mathcal{G}$ contains at least one of v_1, v_2 or v_3. Since all of these vertices see v_1, $v_1 \in \textsc{vis}\, \mathcal{G}$. Symmetrically, we can prove that $\mathcal{G}$ sees v_n if $v_n \in \mathscr{R}_r$. Thus, if $v_i \in \mathscr{R}$, $v_i \in \textsc{vis}\, \mathcal{G}$. This implies that $\textsc{vis}\, \mathcal{G} \supseteq \mathscr{R}$. Since $V = \mathscr{C} \cup \mathscr{R}$, $\textsc{vis}\, \mathcal{G} \supseteq V$. □

Here, we have proved that if we ensure that the convex vertices of an orthogonal terrain are guarded then all the vertices of that terrain are guarded. We strengthen this further by proving that the set of convex vertices can be guarded by k guards placed anywhere on the terrain if, and only if, there exists a set of guards placed on the reflex vertices of size at most k which sees all the convex vertices in the following lemma.

Lemma 2.8. *Let $T(V, E)$ be an orthogonal terrain and $\mathcal{G} \subset \mathrm{Im}\, T$ be a finite set of guards such that $\textsc{vis}\, \mathcal{G} \supseteq \mathscr{C}$. Then, there exists a $\mathcal{G}' \subseteq \mathscr{R}$ with $|\mathcal{G}'| \leq |\mathcal{G}|$ such that $\textsc{vis}\, \mathcal{G}' \supseteq \mathscr{C}$.*

Proof. We construct a set $\mathcal{G}' \subseteq \mathscr{R}$ of guards as follows. Consider $g \in \mathcal{G}$. Assume g lies in the interior of a horizontal edge (v_i, v_{i+1}). Then, for any $w \in \textsc{vis}\, g \cap \mathscr{C}$, $y(w) = y(g)$. Thus, v_i and v_{i+1} see w. If $v_i \in \mathscr{R}$, place a guard g' at v_i. If $v_i \in \mathscr{C}$ and $i = 1$, place a guard g' at v_{i+1} (that is, at v_2). If $v_i \in \mathscr{C}$ and $i \neq 1$, then $v_{i-1} \in \mathscr{R}$. Also, v_{i-1} lies above $\overline{v_i w}$ and $x(v_{i-1}) = x(v_i)$. Thus, $w \in \textsc{vis}\, v_{i-1}$. Place a guard g' at v_{i-1}. Add g' to $\mathcal{G}'$. By construction, $|\mathcal{G}'| \leq \mathcal{G}$ and $\textsc{vis}\, \mathcal{G}' \supseteq \mathscr{C}$. □

Succinctly put, these two lemmas give us the following theorem.

Theorem 2.9. $\overline{(T(V, E), n, k)} \equiv (T(V, E), n, k) \equiv (T(V, E), n, k, \mathscr{R}, \mathscr{C})$ *if $T(V, E)$ is an orthogonal terrain.*

We now establish the relationship between the terrain guarding problem in orthogonal terrains and the minimum clique cover problem in chordal graphs. Consider a Discrete Terrain Guarding instance $(T(V, E), n, k)$ where T is an orthogonal terrain. By Theorem 2.9, V can be guarded by k guards if, and only if, $\mathscr{C}$ can be guarded by $\mathcal{G} \subseteq \mathscr{R}$, where $|\mathcal{G}| \leq k$. Now, since $\mathscr{C} = \mathscr{C}_l \cup \mathscr{C}_r$, $\mathscr{C}$ can be guarded by at most k guards if, and only if, $\mathscr{C}_l$ and $\mathscr{C}_r$ can be guarded by k_l and k_r guards, say $\mathcal{G}_l \subseteq \mathscr{R}$ and $\mathcal{G}_r \subseteq \mathscr{R}$, where $k_l + k_r = k$.

Observation 2.6 tells us that $\mathcal{G}_l$ can be considered to be a set of left guards and $\mathcal{G}_r$ can be considered to be the of right guards. Consider the Annotated Terrain Guarding instance $(T(V,E), n, k_l, \mathcal{G}_l, \mathscr{C}_l)$. Let $G_{T(l)}$ be the visibility graph of this instance. Since $\mathcal{G}_l \subseteq \mathscr{R}$ and $\mathscr{C}_l \cap \mathscr{R} = \emptyset$ by definition, $\mathcal{G}_l \cap \mathscr{C}_l = \emptyset$. Thus, by Observation 2.4, $\mathcal{G}_l$ can guard $\mathscr{C}_l$ if, and only if, $G_{T(l)}$ can be covered by at most k_l cliques. Similarly, $\mathcal{G}_r$ can guard $\mathscr{C}_r$ if, and only if, $G_{T(r)}$ can be covered by at most k_r cliques where $G_{T(r)}$ is the visibility graph of the instance $(T(V,E), n, k_r, \mathcal{G}_r, \mathscr{C}_r)$. By Theorem 2.2, $G_{T(l)}$ and $G_{T(r)}$ are chordal. This gives us the following lemma.

Lemma 2.10. *A* Discrete Terrain Guarding *instance $(T(V,E), n, k)$ where T is an orthogonal terrain is a* Yes *instance if, and only if, there exists $k_l, k_r \in \mathbb{N}$, $k_l + k_r = k$ such that both the* Clique Cover *instances $(G_{T(l)}, |\mathscr{C}_l|, k_l)$ and $(G_{T(r)}, |\mathscr{C}_r|, k_r)$ are* Yes *instances. Furthermore, $G_{T(l)}$ and $G_{T(r)}$ are chordal graphs.*

The rest of the section will be used to develop and describe polynomial time algorithms to decide a Clique Cover instance $(G(V,E), n, k)$ and to solve the maximum sized independent set problem in chordal graphs. The algorithms that follow are adaptations of those found in [6]. The variables used in the rest of the section have no intrinsic relationship with those in the terrain guarding instances described so far. We first define simplicial vertices and a perfect elimination order. We then prove that chordal graphs have a perfect elimination order. Using this, we will describe a greedy algorithm to decide a Clique Cover instance. We let $N_G(v)$, for any vertex v in a graph G, to denote the neighbourhood of v in G and $N_G[v]$ be $N_G(v) \cup \{v\}$.

Definition. Let $G(V,E)$ be a graph. A vertex $v \in V$ is said to be *simplicial* if $N_G(v)$ is a clique.

Definition. Let $G(V,E)$ be a graph and σ be a permutation of the set V. σ is a *perfect elimination ordering* if the set $W_i = \{v_j \mid v_j \in N_G(v_i), j > i\}^\dagger$ is a clique for all $v_i \in V$.

A perfect elimination ordering is abbreviated as a PEO. For the definition and the observation that follows, let $G(V,E)$ be a graph which is not complete and $a, b \in V$ be two non-adjacent vertices in G.

Definition. $S \subset V$ is a *(a,b)-separator* if a and b are not connected in $G[V \setminus S]$. S is a *minimal (a,b)-separator* if no proper subset of S has this property.

Observation 2.11. Consider $s \in S$ where S is a minimal (a,b)-separator. Then, there exists $s_a \in V_a$, $s_b \in V_b$ such that s_a, $s_b \in N_G(s)$ where V_a and V_b denote the connected components containing a and b in $G[V \setminus S]$ respectively.

Since $V \setminus \{a,b\}$ is an (a,b)-separator, a minimal (a,b)-separator always exists.

Theorem 2.12. *Every chordal graph $G(V,E)$ has a simplicial vertex. If G is not complete, then G has two non-adjacent simplicial vertices.*

Proof. Let $G(V,E)$ be a chordal graph. If G is complete, any vertex $v \in V$ is simplicial. Assume G is not complete. We prove that G has two non-adjacent simplicial vertices by induction on $|V|$. The base case

†Some texts define $W_i = \{v_j \mid v_j \in N_G(v_i), j < i\}$.

is the graph $G(V,E)$ where $V = \{a,b\}$ and E is empty. Clearly, a,b are simplicial. Now, assume that the supposition is true for all graphs which are not complete whose vertex set has at most n elements. Consider $G(V,E)$ where $|V| = n$ and $a,b \in V$ are non-adjacent vertices. Let S be a minimal (a,b)-separator.

We first prove that S is a clique. Consider $x,y \in S$. By Observation 2.11, there exists a path between x and y whose vertices are from V_a except for the endpoints themselves. Let the smallest such path be p_a. Similarly, let the smallest path from x to y through V_b be p_b. If either of p_a or p_b are $\{x,y\}$, then $(x,y) \in E$. Assume otherwise. Then $p_a \cup p_b$ forms a cycle of length at least 4. Since G is chordal, by Observation 1.1, $G[V_a \cup S \cup V_b]$ is chordal. Thus, there exists an edge between two vertices in $p_a \cup p_b$, say (e,f), which is not part of the edges describing the cycle. Since S is a separator, both e and f are in either p_a or p_b. Assume, without loss in generality, that $e,f \in p_a$ and that e appears before f in its description. Then, the path described as follows: from x to e as in p_a, e to f through this edge and from f to y as in p_a, is smaller than p_a. This is a contradiction to our assumption that p_a is the smallest edge from x to y through V_a. Thus, S is a clique.

If $G[V_a \cup S]$ is complete, then any vertex in V_a is simplicial in $G[V_a \cup S]$. Now assume that $G[V_a \cup S]$ is not complete. Since $b \notin V_a \cup S$, the cardinality of $V_a \cup S$ is strictly less than n. By induction hypothesis, there exists two non-adjacent simplicial vertices in $G[V_a \cup S]$. Since S is a clique, one of these vertices must be in V_a. Thus, in both cases, there exists a vertex, say $s_a \in V_a$ which is simplicial in $G[V_a \cup S]$. Since $N_G(s_a) \subseteq V_a \cup S$, s_a is simplicial in G. Similarly, there exists $s_b \in V_b$ which is simplicial in G. Since S is separator, s_a and s_b are non-adjacent. This completes the proof by induction. □

Observation 2.13. Let $G(V,E)$ be a graph and $\alpha(G)$ be the size of a maximum independent set in G and $\overline{\chi}(G)$ be the size of a minimum clique cover of G. Then, $\alpha(G) \leq \overline{\chi}(G)$.

We now describe a few simple algorithms regarding chordal graphs.

Algorithm 2: Computing a PEO in Chordal Graphs

```
Input  : A chordal graph G(V,E).
Output: Perfect elimination ordering of G.
n ← |V|
G′(V′,E′) ← G(V,E)
σ[ ]
for i ← 0 to n − 1 do
    for v ∈ V′ do
        if v is simplicial then
            σ_i ← v
            G′(V′,E′) ← G′[V′ \ {v}]
            break for
        end if
    end for
    i ← i + 1
end for
return σ
```

Lemma 2.14. *Given a chordal graph $G(V,E)$,* Algorithm 2 *runs in $\mathcal{O}(|V|^2|E|)$ time to compute a PEO of G.*

Proof. By Observation 1.1, $G'(V', E')$ is chordal after each iteration of the loop. By Theorem 2.12, there exists a simplicial vertex in G'. Thus, σ is a perfect elimination ordering by construction. $\square$

Algorithm 3: Clique Cover Problem in Graphs With a PEO

Input : A CLIQUE COVER instance $(G(V, E), n, k)$ and its PEO $\sigma[\,]$.
Output: YES if, and only if, G can be covered by k cliques.

```
G'(V', E') ← G(V, E)
σ'[ ] ← σ[ ]
i ← 0
y[ ]
K[ ]
while σ'[ ] is not empty do
    y_i ← σ'_0
    K_i ← N_{G'}[y_i]
    DELETE K_i from σ'[ ]
    G'(V', E') ← G'[V' \ K_i]
    i ← i + 1
end while
χ̄' ← |K[ ]|
if χ̄' ≤ k then
    return YES
end if
else
    return NO
end if
```

Lemma 2.15. *Given a* CLIQUE COVER *instance* $(G(V, E), n, k)$ *and its PEO* σ*,* ALGORITHM 3 *runs in* $\mathcal{O}(|V| + |E|)$ *time and returns* YES *if, and only if, the* CLIQUE COVER *instance is a* YES *instance.*

Proof. The set $\{y_i \mid 0 \leq i < \overline{\chi}'\}$ is an independent set by construction and has cardinality $\overline{\chi}'$. Thus, $\alpha(G) \geq \overline{\chi}'$. We first prove that K is a clique cover of G. $y_i = v_{i^*}$ for some $v_{i^*} \in \sigma$. Since y_i is taken to be the first element of σ', $N_{G'}(y_i) = N_{G'}(v_{i^*}) \subseteq W_i$ where $W_i = \{v_j \mid v_j \in N_G(v_{i^*}), j > i^*\}$. Since σ is a PEO, W_i is a clique for each i. Thus, $N_{G'}(y_i)$ is a clique. This implies that $K_i = N_{G'}[y_i]$ is a clique for each i. Thus, by construction, $K = \{K_i \mid 0 \leq i < \overline{\chi}'\}$ is a clique cover of G.

To prove that K is the minimal clique cover of G we now prove that $|K| = \overline{\chi}' = \overline{\chi}(G)$. Since K is a clique cover, $\overline{\chi}' \geq \overline{\chi}(G)$. This implies that $\alpha(G) \geq \overline{\chi}'(G) \geq \overline{\chi}(G)$. By Observation 2.13, $\alpha(G) = \overline{\chi}'(G) = \overline{\chi}(G)$. Thus, the minimum number of cliques required to cover G is $\overline{\chi}'(G)$ and a CLIQUE COVER instance $(G(V, E), n, k)$ is YES if, and only if, $\overline{\chi}'(G) \leq k$. $\square$

Observation 2.16. Given a chordal graph $G(V, E)$, on applying ALGORITHM 2 followed by ALGORITHM 3, we solve a CLIQUE COVER instance $(G(V, E), n, k)$ in $\mathcal{O}(|V|^2|E|)$ time.

The proof of Lemma 2.15 gives us the following observation and an algorithm running in $\mathcal{O}(|V|^2|E|)$ time which returns the maximum independent set of a chordal graph given its PEO.

Observation 2.17. Let $G(V, E)$ be a chordal graph. Then, $\alpha(G) = \overline{\chi}(G)$.

Algorithm 4: Independent Set in Graphs With a PEO

Input : A chordal graph $(G(V, E)$.
Output: A maximum independent set I of G.
$G'(V', E') \leftarrow G(V, E)$
$\sigma'[\,] \leftarrow$ ALGORITHM 2$(G(V, E))$
$i \leftarrow 0$
$y[\,]$
while $\sigma'[\,]$ *is not empty* **do**
  $y_i \leftarrow \sigma'_0$
  DELETE $N_{G'}[y_i]$ from $\sigma'[\,]$
  $G'(V', E') \leftarrow G'[V' \setminus N_{G'}[y_i]]$
  $i \leftarrow i + 1$
end while
$I[\,] \leftarrow y[\,]$
return $I[\,]$

3 Algorithms for Terrain Guarding

In this section, we will first describe an algorithm to guard a terrain using only left or right guards using which we design a 2-approximation algorithm for orthogonal terrains. We will conclude the section by looking at a FPT algorithm with parameter k to decide a DISCRETE TERRAIN GUARDING instance $(T(V, E), n, k)$ where T is an orthogonal terrain.

Algorithm 5: Guarding a Terrain with Left or Right Guards

Input : An ANNOTATED TERRAIN GUARDING instance $(T(V, E), n, k, \mathcal{G}, \mathcal{C})$ where $\mathcal{G}$ is a set of left or right guards and $\mathcal{G} \cap \mathcal{C} = \emptyset$.
Output: YES if, and only if, k guards can guard $\mathcal{C}$.
$p \leftarrow |\mathcal{C}|$
$G_T(\mathcal{C}, E') \leftarrow$ ALGORITHM 1 $(T(V, E), n, k, \mathcal{G}, \mathcal{C})$
$\sigma \leftarrow$ ALGORITHM 2 $(G_T(\mathcal{C}, E'))$
return ALGORITHM 3 $((G_T(\mathcal{C}, E'), p, k), \sigma)$

Lemma 3.1. ALGORITHM 5*, given an* ANNOTATED TERRAIN GUARDING *instance* $(T(V, E), n, k, \mathcal{G}, \mathcal{C})$ *where* $\mathcal{G}$ *is a set of left or right guards and* $\mathcal{G} \cap \mathcal{C} = \emptyset$*, runs in* $\mathcal{O}(|V|^4)$ *time and returns a* YES *instance if, and only if, the* ANNOTATED TERRAIN GUARDING *is a* YES *instance.*

Proof. By Theorem 2.2, $G_T(\mathcal{C}, E')$ is chordal. Thus, by Observation 2.16, ALGORITHM 5 decides the CLIQUE COVER INSTANCE $(G_T(\mathcal{C}, E'), p, k)$. By Observation 2.4, $(T(V, E), n, k, \mathcal{G}, \mathcal{C})$ is a YES instance if, and only if, $(G_T(\mathcal{C}, E'), p, k)$ is a YES instance. □

Lemma 3.2. ALGORITHM 6*, given an* ANNOTATED TERRAIN GUARDING *instance* $(T(V, E), n, k, \mathcal{G}, \mathcal{C})$ *where* $\mathcal{G}$ *is a set of left or right guards and* $\mathcal{G} \cap \mathcal{C} = \emptyset$*, runs in* $\mathcal{O}(|V|^4)$ *time and returns an optimal guard set which guards* $\mathcal{C}$*.*

Proof. By Theorem 2.2, $G_T(\mathcal{C}, E')$ is chordal. Also, by the proof of correctness of ALGORITHM 3, K is an optimal clique cover of G_T. We will now prove that, for any $K_j \in K$, $\text{VIS}\,\mathcal{G}'_j \supseteq K_j$. Let $|K_j| = m$. We

Algorithm 6: Set of Left or Right Guards to Guard a Terrain

Input : An ANNOTATED TERRAIN GUARDING instance $(T(V,E), n, k, \mathcal{G}, \mathcal{C})$ where $\mathcal{G}$ is a set of left or right guards and $\mathcal{G} \cap \mathcal{C} = \emptyset$.

Output: An optimal set of guards, $\mathcal{G}' \subseteq \mathcal{G}'$, such $\textsc{vis}\,\mathcal{G}' \supseteq \mathcal{C}$.

$E'[\ ]$
$Guards[\ ]$
for *each distinct pair of vertices* $(v_i, v_j) \in \mathcal{C}^2$, $i < j$ **do**
  if *there is a* $g \in \mathcal{G}$ *seeing* v_i *and* v_j **then**
    ADD (v_i, v_j) to $E'[\ ]$
    $Guards_{(i,j)} \leftarrow g$
  end if
end for
$y[\ ]$
$K[\ ]$
$\sigma[\ ] \leftarrow$ ALGORITHM 2 $(G_T(\mathcal{C}, E'))$
$i \leftarrow 0$
while $\sigma[\ ]$ *is not empty* **do**
  $y_i \leftarrow \sigma_0$
  $K_i \leftarrow N_{G_T}[y_i]$
  DELETE K_i from $\sigma'[\ ]$
  $G_T(\mathcal{C}, E') \leftarrow G_T[\mathcal{C} \setminus K_i]$
  $i \leftarrow i + 1$
end while
$\mathcal{G}'[\ ]$
for $K_j \in K[\ ]$ **do**
  $m \leftarrow |K_j|$
  SORT vertices of K_j according to their position on the terrain.
  Assume $\{v_{k_0}, v_{k_1} \dots v_{k_{m-1}}\}$ is the sorted order of K_j.
  $\mathcal{G}'_j \leftarrow Guards_{(k_{m-2}, k_{m-1})}$
  $l \leftarrow m - 3$
  while $l \geq 0$ **do**
    if $\mathcal{G}'_j \prec Guards_{(k_l, k_{m-1})}$ **then**
      $\mathcal{G}'_j \leftarrow Guards_{(k_l, k_{m-1})}$
    end if
    $l \leftarrow l - 1$
  end while
end for
return $\mathcal{G}'[\ ]$

prove, by induction, that, after i iterations of the **while** loop, $\textsc{vis}\,\mathcal{G}'_j \supseteq \{v_{k_{m-i-2}}, v_{k_{m-i-1}} \dots v_{k_{m-1}}\}$, where $0 \le i \le m-2$. Let $l_i = m-i-2$ and $K^i_j = \{v_{k_{m-i-2}}, v_{k_{m-i-1}} \dots v_{k_{m-1}}\}$. Since $\mathcal{G}'_j$ is initialized to be the guard seeing both $v_{k_{m-1}}$ and $v_{k_{m-2}}$, the base case is true. Now, assume that for all $i < m_0$ where $m_0 \in \mathbb{N}$ and $m_0 \le m-2$, our supposition is true. Thus, $\textsc{vis}\,\mathcal{G}'_j \supseteq K^{m_0-1}_j$. By the proof of Theorem 2.3, $Guards_{(k_{l_{m_0}}, k_{m-1})}$ sees $K^{m_0}_j$ if $\mathcal{G}'_j \prec Guards_{(k_l, k_{m-1})}$ while $\mathcal{G}'_j$ sees the set otherwise. Thus, by construction, after m_0 iterations, $\textsc{vis}\,\mathcal{G}'_j \supseteq K^{m_0}_j$ proving our supposition. Thus, after the completion of the **while** loop, $\textsc{vis}\,\mathcal{G}'_j \supseteq K_j$. Since K is an optimal clique cover, $\mathcal{G}'$ is an optimal guard set which sees $\mathcal{C}$. □

Algorithm 7: 2-Approximation Algorithm to Guard an Orthogonal Terrain

Input : An orthogonal terrain $T(V,E)$.
Output: A set of guards, $\mathcal{G}'$ with $\textsc{vis}\,\mathcal{G}' \supseteq V$ and $|\mathcal{G}'| \le 2 \cdot \textsc{opt}$ where $\textsc{opt}$ is the minimum number of guards required to guard the terrain.

$n \leftarrow |V|$
$\mathcal{G}_l[\,] \leftarrow$ Algorithm 6 $(T(V,E), n, n, \mathscr{R}, \mathscr{C}_l)$
$\mathcal{G}_r[\,] \leftarrow$ Algorithm 6 $(T(V,E), n, n, \mathscr{R}, \mathscr{C}_r)$
$\mathcal{G}'[\,] \leftarrow \mathcal{G}_l \cup \mathcal{G}_r$
return $\mathcal{G}'[\,]$

Lemma 3.3. *Given an orthogonal terrain $T(V,E)$,* Algorithm 7*, running in $\mathcal{O}(|V|^4)$ time, returns a set of guards, $\mathcal{G}'$ with $\textsc{vis}\,\mathcal{G}' \supseteq V$ and $|\mathcal{G}'| \le 2 \cdot \textsc{opt}$ where $\textsc{opt}$ is the minimum number of guards required to guard the terrain.*

Proof. By Observation 2.6, $\mathcal{C}_l$ can be guarded by left guards while $\mathcal{C}_r$ can be guarded by right guards. Furthermore, by Theorem 2.9, these guards can be placed in the reflex vertices of the terrain. Thus, $\mathcal{G}_l$ and $\mathcal{G}_r$ have at most $\textsc{opt}$ guards each. Since $\mathcal{G}' = \mathcal{G}_l \cup \mathcal{G}_r$, $|\mathcal{G}'| \le 2 \cdot \textsc{opt}$ and $\textsc{vis}\,\mathcal{G}' \supseteq \mathscr{C}$. By Theorem 2.9, $\textsc{vis}\,\mathcal{G}' \supseteq V$. Also, since $|\mathscr{C}|, |\mathscr{R}| \in \mathcal{O}(|V|)$, the running time of this algorithm is $\mathcal{O}(|V|^4)$. □

We will now define a few terms and prove some lemmas that will help us design the final algorithm of this report. The rest of this section is based on [1].

Definition. In parameterized complexity, for a problem $\mathcal{P}$, an instance I is associated with a real number, say k. k is called the *parameter* of $\mathcal{P}$. $\mathcal{P}$ is said to be *fixed parameter tractable (FPT)*, with respect to a parameter, if any instance I can be solved in $\mathcal{O}(f(k) \cdot |I|^{\mathcal{O}(1)})$ time where f is some computable function independent of $|I|$.

We will prove that a Discrete Terrain Guarding instance $(T(V,E), n, k)$ where T is an orthogonal terrain is FPT with respect to k by describing an algorithm that decides such an instance in $k^{\mathcal{O}(k)} \cdot n^{\mathcal{O}(1)}$ time.

Lemma 3.4. *Consider an orthogonal terrain $T(V,E)$. Let $U \subseteq \mathscr{C}_l$ and $k' \in \mathbb{N}$. Then, one can decide in polynomial time if there exists a $S \subseteq \mathscr{R}_l$ of size at most k' and $\textsc{vis}\,S \supseteq U$. If such a set does not exist, then, in polynomial time, one can find a $U' \subseteq U$ whose size is $k'+1$ such that there does not exist any set $S \subseteq \mathscr{R}_l$ of size k' which sees all of U'. A symmetric claim holds for a subset of $\mathscr{C}_r$ vertices.*

Proof. Given the instance $(T(V,E), n, k', \mathscr{R}, U)$, ALGORITHM 5 decides, in polynomial time, if it is an YES instance. This proves the first part of the lemma. If the instance is NO, then, by Observation 2.4, the size of the minimum clique cover of $G_T(U, E')$ is strictly greater than k'. By Observation 2.17, $|I| > k'$ where I is the independent set returned by ALGORITHM 4 $(G_T(U, E'))$. Consider $U' \subseteq I$ where $|U'| = k' + 1$. If some k' guards see all of U', then one of these guards must see two vertices in U'. This contradicts the assumption that U' is an independent set. Since ALGORITHM 4 runs in polynomial time, U' can be found in polynomial time. The proof follows symmetrically for a subset of $\mathscr{C}_r$ vertices. $\square$

Definition. Let $(T(V,E), n, k)$ be a DISCRETE TERRAIN GUARDING instance where T is an orthogonal terrain. A vertex $e \in \mathscr{C}$ is defined to be *exposed* if it is seen by more than $k+2$ many vertices of the opposite type.

In Figure 10, e is an exposed vertex when $k = 2$. Let $\mathcal{E} \subseteq \mathscr{C}$ be the set of exposed vertices. We let $\mathcal{E}_l = \mathcal{E} \cap \mathscr{C}_l$ and $\mathcal{E}_r = \mathcal{E} \cap \mathscr{C}_r$. Let $e \in \mathcal{E}_l$. If $e \neq v_1$, we let u_1^e, $u_2^e \ldots u_{k+3}^e$ denote the $k+3$ leftmost vertices in $\mathscr{R}_r$ which see e sorted from left to right by the order in which they lie on the terrain. If e is v_1 we let u_2^e, $u_3^e \ldots u_{k+3}^e$ denote the $k+2$ leftmost vertices in $\mathscr{R}_r$ which see e which are sorted as before. Similarly, given $e \in \mathcal{E}_r$ and $e \neq v_n$, we let u_1^e, $u_2^e \ldots u_{k+3}^e$ denote the $k+3$ rightmost vertices in $\mathscr{R}_l$ which see e sorted from right to left by the order in which they lie on T. If $e = v_n$, we have u_2^e, $u_3^e \ldots u_{k+3}^e$ to denote the $k+2$ rightmost vertices that see e. They are sorted as before. We prove, using the following observations and lemmas, that $(T(V,E), n, k, \mathscr{R}, \mathscr{C}) \equiv (T(V,E), n, k, \mathscr{R}, \mathscr{C} \setminus \mathcal{E})$ for orthogonal terrains.

Observation 3.5. Consider $e \in \mathcal{E}_l$. Then,

(i) $x(u_1^e) = x(e)$, $y(u_1^e) > y(e)$ if u_1^e is defined.

(ii) $x(u_i^e) > x(e)$, $y(u_i^e) = y(e)$ for all $2 \leq i \leq k+3$.

Similarly, if $e \in \mathcal{E}_r$. Then,

(i) $x(u_1^e) = x(e)$, $y(u_1^e) > y(e)$ if u_1^e is defined.

(ii) $x(u_i^e) < x(e)$, $y(u_i^e) = y(e)$ for all $2 \leq i \leq k+3$.

Observation 3.6. Let $e \in \mathcal{E}$. Then, for any vertex $v \in \mathscr{C}$ which lies between e and u_{k+3}^e, $y(v) < y(e)$.

Observation 3.7. Let $e \in \mathcal{E}$. Consider indices i and j such that $2 \leq i, j \leq k+2$ and $i \neq j$. Then, no vertex between u_i^e and u_{i+1}^e sees any vertex between u_j^e and u_{j+1}^e.

Lemma 3.8. *Let $(T(V,E), n, k)$ be a* DISCRETE TERRAIN GUARDING *instance where T is an orthogonal terrain. Consider an exposed vertex $e \in \mathcal{E}_l$. Then, for any $2 \leq i \leq k+2$, there exists $w \in \mathscr{C}_l \setminus \mathcal{E}_l$ such that $u_i^e \prec w \prec u_{i+1}^e$. Symmetrically, for an exposed vertex $e \in \mathcal{E}_r$, for any $2 \leq i \leq k+2$, there exists $w \in \mathscr{C}_r \setminus \mathcal{E}_r$ such that $u_{i+1}^e \prec w \prec u_i^e$.*

Proof. Consider an exposed vertex $e \in \mathcal{E}_l$ and $2 \leq i \leq k+2$. Since $u_i^e \in \mathscr{R}_r$, the vertex following it in the terrain is a left convex vertex. Thus, there exists a vertex $v \in \mathscr{C}_l$ such that $u_i^e \prec v \prec u_{i+1}^e$. Let $w \in \mathscr{C}_l$ be the vertex between u_i^e and u_{i+1}^e with the least y-coordinate. If $w \in \mathcal{E}_l$, then there exists right reflex vertices u_2^w and u_3^w which see w. By the same argument as above, there exists a vertex $w' \in \mathscr{C}_l$ such that $u_1^w \prec w' \prec u_2^w$.

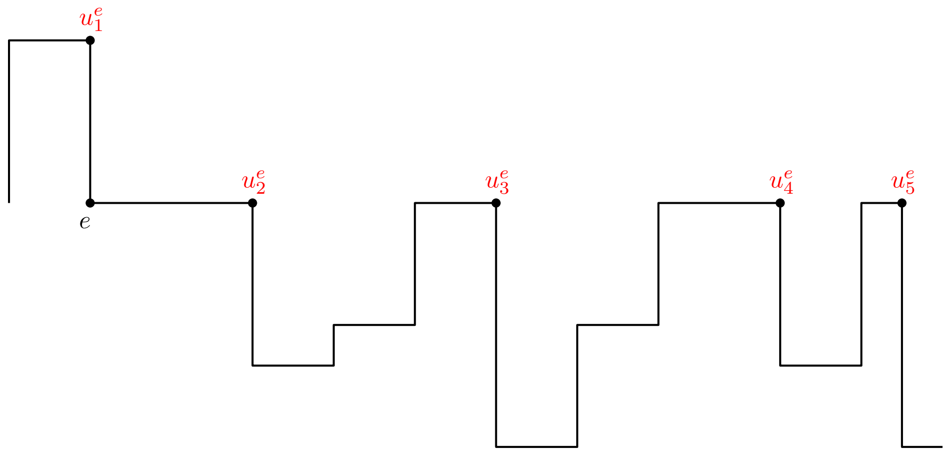


Figure 10: The vertex e is exposed when we take $k = 2$.

By Observation 3.6, $w'_y < w_y$. By Observation 3.7, u_2^w and u_3^w lie in between u_i^e and u_{i+1}^e. Thus, w' is a left convex vertex which lies between u_i^e and u_{i+1}^e whose y-coordinate is strictly less than that of w. This is a contradiction to the construction of w. This proves the first part of the lemma. The proof of the second part of the lemma follows symmetrically. $\square$

Lemma 3.9. *Let $T(V, E)$ be an orthogonal terrain and $(T(V, E), n, k, \mathscr{R}, \mathscr{C} \setminus \mathcal{E})$ be a* Yes *instance of the* Discrete Terrain Guarding *problem. Let $\mathcal{G}' \subseteq \mathscr{R}$ be a set of guards such that $\text{vis}\,\mathcal{G}' \supseteq \mathscr{C} \setminus \mathcal{E}$ and $|\mathcal{G}'| \leq k$. Consider a vertex $e \in \mathcal{E}_l$. Then, there exists an index $2 \leq i \leq k + 2$ such that a vertex between u_i^e and u_{i+1}^e is seen by a $g \in \mathcal{G}'$ which lies to the right of u_{i+1}^e. Symmetrically, for a vertex $e \in \mathcal{E}_r$, there exists an index $2 \leq i \leq k + 2$ such that a vertex between u_i^e and u_{i+1}^e is seen by a $g \in \mathcal{G}'$ which lies to the left of u_{i+1}^e.*

Proof. Consider an exposed vertex $e \in \mathcal{E}_l$. By Lemma 3.8, for all $2 \leq i \leq k + 2$, there exists a $w_i \in \mathscr{C}_l \setminus \mathcal{E}_l$. Since $|\{w_i \mid 2 \leq i \leq k + 2\}| = k + 1$ and $|\mathcal{G}'| \leq k$, by Observation 3.7, there exists an index j, $2 \leq j \leq k + 2$ such that w_j is seen by a guard $g \in \mathcal{G}'$ which is not between u_j^e and u_{j+1}^e. By Observation 2.6, g lies to the right of w_j. Thus, g lies to the right of u_{j+1}^e. The proof of the second part of the lemma follows symmetrically. $\square$

Lemma 3.10. *$(T(V, E), n, k, \mathscr{R}, \mathscr{C}) \equiv (T(V, E), n, k, \mathscr{R}, \mathscr{C} \setminus \mathcal{E})$ where T is an orthogonal terrain.*

Proof. Assume that $(T(V, E), n, k, \mathscr{R}, \mathscr{C})$ is a Yes instance. Since $\mathscr{C} \setminus \mathcal{E} \subseteq \mathscr{C}$, $(T(V, E), n, k, \mathscr{R}, \mathscr{C} \setminus \mathcal{E})$ is a Yes instance. Now, assume that there exists $\mathcal{G}' \subseteq \mathscr{R}$, a set of guards, such that $\text{vis}\,\mathcal{G}' \supseteq \mathscr{C} \setminus \mathcal{E}$ and $|\mathcal{G}'| \leq k$. Consider an vertex $e \in \mathcal{E}_l$. By Lemma 3.9, there exists an index j where $2 \leq j \leq k + 2$ such that $w_j \in \mathscr{C}_l \setminus \mathcal{E}_l$ is seen by a guard $g \in \mathcal{G}'$ which lies to the right of u_{j+1}^e. This is illustrated in Figure 11. On applying Lemma 2.1, we get that g sees e. Since e was arbitrary, $\text{vis}\,\mathcal{G}' \supseteq \mathcal{E}_l$. Symmetrically, $\text{vis}\,\mathcal{G}' \supseteq \mathcal{E}_r$. Thus, $\text{vis}\,\mathcal{G}' \supseteq \mathscr{C}$. This proves that $(T(V, E), n, k, \mathscr{R}, \mathscr{C})$ is a Yes instance. $\square$

Finally, we will now give a linear bound on the number of non-exposed vertices which are seen by vertices of the opposite type. Given an orthogonal terrain, let $\mathcal{C} \subseteq \mathscr{C} \setminus \mathcal{E}$ and $|\mathcal{C}| = m$. Let $v_1, v_2, \ldots v_m$ denote the vertices in $\mathcal{C}$ such that $v_i \prec v_j$ for all $1 \leq i < j \leq m$. Let $\mathcal{G} \subseteq \mathscr{R}$ such that $\text{vis}\,\mathcal{G} \supseteq \mathcal{C}$. Define $\bar{b}$ as the

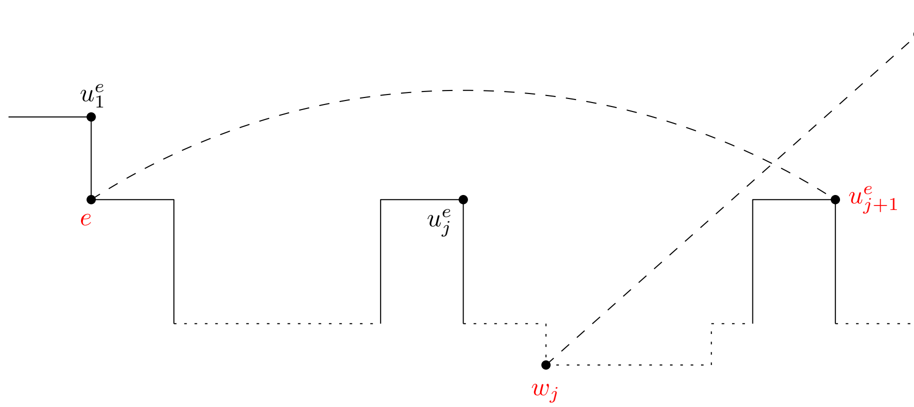


Figure 11

m-length bit vector $(b_1, b_2, \ldots b_m)$, such that for any $1 \leq i \leq m$, $b_i = 0$ if $v_i \in \mathscr{C}_l \setminus \mathcal{E}$ and $b_i = 1$ otherwise. Also, define $\hat{b}$ as the m-length bit vector $(b'_1, b'_2, \ldots b'_m)$, such that for any $1 \leq i \leq m$, $b'_i = 0$ if v_i is seen by a left reflex vertex and $b'_i = 1$ otherwise.

The hamming distance between two vectors $\overline{v}$ and $\overline{w}$ of the length n, denoted by $H(\overline{v}, \overline{w})$, is the number of indices i where $1 \leq i \leq n$ such that $v_i \neq w_i$. $H(\overline{b}, \hat{b})$ is the number of left convex vertices that are only seen by right reflex vertices plus the number of right convex vertices that are seen by a left reflex vertex.

Observation 3.11. Let $v \in \mathscr{R}$. Then, v sees at most two vertices of the opposite type. One of these vertices have the same x-coordinate as v while the other has the same y-coordinate as v.

Lemma 3.12. *Let $T(V, E)$ be an orthogonal terrain and $(T(V, E), n, k, \mathscr{R}, \mathcal{C})$ be a* Yes *instance of the* Discrete Terrain Guarding *problem where $\mathcal{C} \subseteq \mathscr{C} \setminus \mathcal{E}$. Let $\mathcal{G}' \subseteq \mathscr{R}$ be a set of guards such that $\text{VIS}\,\mathcal{G}' \supseteq \mathcal{C}$ and $|\mathcal{G}'| \leq k$. Then, $H(\overline{b}, \hat{b}) \leq 2k$.*

Proof. Since $|\mathcal{G}'| \leq k$, by Observation 3.11, $\mathcal{G}'$ sees at most $2k$ vertices of the opposite type. That is, the number of left convex vertices that are seen by a right reflex vertices plus the number of right convex vertices that are seen by a left reflex vertex is at most $2k$. Thus, $H(\overline{b}, \hat{b}) \leq 2k$. $\square$

Definition. $((T(V, E), n, k, \mathscr{R}, \mathcal{C}), \delta, k_l, k_r)$ is called an *identifiable instance* if there exists $\mathcal{G} \subseteq \mathscr{R}$ where $\text{VIS}\,\mathcal{G} \supseteq \mathcal{C}$ such that $|\mathcal{G} \cap \mathscr{R}_l| \leq k_l$, $|\mathcal{G} \cap \mathscr{R}_r| \leq k_r$, and $H(\overline{b}, \hat{b}) \leq \delta$.

Theorem 3.13. *Let $(T(V, E), n, k, \mathscr{R}, \mathcal{C})$ be a* Discrete Terrain Guarding *instance where T is an orthogonal terrain and $\mathcal{C} \subseteq \mathscr{C} \setminus \mathcal{E}$. Then,* Algorithm 8 *$((T(V, E), n, k, \mathscr{R}, \mathcal{C}), \delta = 2k, k_l, k_r)$ returns* Yes *if, and only if, there exists k_l guards placed on the left reflex vertices and k_r guards placed on the right reflex vertices which can guard $\mathscr{C}$. Furthermore, it runs in $k^{\mathcal{O}(k)} \cdot n^{\mathcal{O}(1)}$ time.*

Proof. First, assume that there exists $\mathcal{G} \subseteq \mathscr{R}$ where $\text{VIS}\,\mathcal{G} \supseteq \mathscr{C} \setminus \mathcal{E}$ such that $|\mathcal{G} \cap \mathscr{R}_l| = k_l$ and $|\mathcal{G} \cap \mathscr{R}_r| = k_r$. Since $\mathcal{C}_l \subseteq \mathscr{C}_l$, by Lemma 3.4, we can decide in polynomial time if there exists a set of guards of size at most k_l which guards $\mathcal{C}_l$ and if not, find $U_l \subseteq \mathcal{C}_l$ of size $k_l + 1$ which cannot be guarded by k_l guards. Assume that $\mathcal{C}_l$ cannot be guarded by k_l many guards. Then, there exists a vertex $v \in U_l$ which is seen by a right reflex vertex. The algorithm finds v by trying all possible vertices in U_l. Since $U_l \subseteq \mathscr{C} \setminus \mathcal{E}$, v is not exposed.

Algorithm 8: A FPT Algorithm to Guard an Orthogonal Terrain

Input : An ANNOTATED TERRAIN GUARDING instance $(T(V,E), n, k, \mathscr{R}, \mathcal{C})$ where T is an orthogonal terrain, δ, k_l - the number of guards to be placed on the left reflex vertices and k_r - the number of guards to be placed on the right reflex vertices.

Output: YES if, and only if, k_l guards placed on the left reflex vertices and k_r guards place on the right reflex vertices can guard $\mathcal{C}$.

$\mathcal{C}_l \leftarrow \mathcal{C} \cap \mathscr{C}_l$
$\mathcal{C}_r \leftarrow \mathcal{C} \cap \mathscr{C}_r$
if $\delta < 0$ **then**
  return NO
end if
else if ALGORITHM 5 $(T(V,E), n, k_l, \mathscr{R}_l, \mathcal{C}_l)$ *is* NO **then**
  $G_T(\mathcal{C}_l, E') \leftarrow$ ALGORITHM 1 $(T(V,E), n, k, \mathscr{R}_l, \mathcal{C}_l)$
  $I_l \leftarrow$ ALGORITHM 4 $(G_T(\mathcal{C}_l, E'))$
  Choose $U_l \subseteq I_l$ such that $|U_l| = k_l + 1$
  for $v \in U_l$ **do**
    for $u \in \textsc{vis}\, v \cap \mathscr{R}_r$ **do**
      ALGORITHM 8 $((T(V,E), n, k, \mathscr{R}, \mathcal{C} \setminus \textsc{vis}\, u), \delta - 1, k_l, k_r - 1)$
    end for
  end for
end if
else if ALGORITHM 5 $(T(V,E), n, k_r, \mathscr{R}_r, \mathcal{C}_r)$ *is* NO **then**
  $G_T(\mathcal{C}_r, E') \leftarrow$ ALGORITHM 1 $(T(V,E), n, k, \mathscr{R}_r, \mathcal{C}_r)$
  $I_r \leftarrow$ ALGORITHM 4 $(G_T(\mathcal{C}_r, E'))$
  Choose $U_r \subseteq I_r$ such that $|U_r| = k_r + 1$
  for $v \in U_r$ **do**
    for $u \in \textsc{vis}\, v \cap \mathscr{R}_l$ **do**
      ALGORITHM 8 $((T(V,E), n, k, \mathscr{R}, \mathcal{C} \setminus \textsc{vis}\, u), \delta - 1, k_l - 1, k_r)$
    end for
  end for
end if
else
  return YES
end if

Thus, there are at most $k + 2$ many right reflex vertices that see v. For each such right reflex, say u, the algorithm places a guard on u and reduces δ and k_r by one while deleting $\textsc{vis}\, u$ from $\mathcal{C}_l$. The algorithm works symmetrically when $\mathcal{C}_r$ cannot be guarded by k_r many guards. Since there exists at most $2k$ many (v, u) pairs by Lemma 3.12, it tries every such pair and must have made the correct choice in at least one of the paths of it's decision tree. Thus, the algorithm detects that $\mathcal{C}_l$ and $\mathcal{C}_r$ can be guarded by k_l and k_r many guards respectively and returns a Yes instance.

Now, assume that Algorithm 8 $((T(V, E), n, k, \mathscr{R}, \mathscr{C} \setminus \mathcal{E}), \delta = 2k, k_l, k_r)$ returns a Yes. Then, we prove, by induction on δ that $\mathscr{C}$ can be guarded by placing k_l and k_r many guards on the left and right reflex vertices respectively. For the base case, assume that $\delta = 0$. Then, both Algorithm 5 $(T(V, E), n, k_l, \mathscr{R}_l, \mathcal{C}_l)$ and Algorithm 5 $(T(V, E), n, k_r, \mathscr{R}_r, \mathcal{C}_r)$ are Yes instances. This proves that our supposition is true. Now, assume that for all $\delta < d$, where $1 \leq d \leq 2k$, $d \in \mathbb{N}$, our supposition is true. Consider an input to the algorithm that has $\delta = d$. If both Algorithm 5 $(T(V, E), n, k_l, \mathscr{R}_l, \mathcal{C}_l)$ and Algorithm 5 $(T(V, E), n, k_r, \mathscr{R}_r, \mathcal{C}_r)$ are Yes instances, then our supposition is true. If $\mathcal{C}_l$ cannot be guarded by k_l many guards placed on left reflex vertices, then, the algorithm computes $U_l \subseteq \mathcal{C}_l$ of size $k_l + 1$ which cannot be guarded by k_l vertices placed on $\mathscr{R}_l$. Then, since it returns Yes, there exists a $v \in U_l$ and a $u \in \textsc{vis}\, u \cap \mathscr{R}_r$ such that Algorithm 8 $((T(V, E), n, k, \mathscr{R}, \mathcal{C} \setminus \textsc{vis}\, u), \delta - 1, k_l, k_r - 1)$ is a Yes instance. By induction hypothesis, $\mathcal{C} \setminus \textsc{vis}\, u$ can be guarded by placing k_l and $k_r - 1$ many guards on the left and right reflex vertices respectively. Thus, $\mathcal{C}$ can be guarded by placing k_l and k_r many guards on the left and right reflex vertices respectively by including the right reflex vertex u in the guard set of $\mathcal{C} \setminus \textsc{vis}\, u$. The proof for the case where $\mathcal{C}_r$ cannot be guarded by k_r many guards placed on left reflex vertices is symmetric. This proves our supposition and completes the proof by induction.

Finally, since U_l and U_r are of size $\mathcal{O}(k)$ and each vertex in U_l or U_r can be seen by $\mathcal{O}(k)$ many vertices of the opposite type, there are $k^{\mathcal{O}(k)}$ many recursive calls of the algorithm. Since Algorithm 5, Algorithm 1 and Algorithm 4 all run in polynomial time, for each recursive call, $\mathcal{O}(1)$ time is spent. Thus, the overall runtime of the algorithm is $k^{\mathcal{O}(k)} \cdot n^{\mathcal{O}(1)}$. □

Algorithm 9: A FPT Algorithm to Guard an Orthogonal Terrain

Input : A Discrete Terrain Guarding instance $(T(V, E), n, k)$ where T is an orthogonal terrain.
Output: Yes if, and only if, $(T(V, E), n, k)$ is a Yes instance.

```
Guarded ← 0
for k_l ← 0 to k do
    k_r ← k − k_l
    if Algorithm 8 ((T(V,E), n, k, ℛ, 𝒞), 2k, k_l, k_r) is Yes then
        Guarded ← 1
    end if
end for
if Guarded is 1 then
    return Yes
end if
else
    return No
end if
```

Observation 3.14. ALGORITHM 9, given an DISCRETE TERRAIN GUARDING instance $(T(V, E), n, k)$, where T is an orthogonal terrain, runs in $k^{\mathcal{O}(k)} \cdot n^{\mathcal{O}(1)}$ time and returns a YES if, and only if, the instance is a YES instance.

Observation 3.15. Both the DISCRETE TERRAIN GUARDING and CONTINUOUS TERRAIN GUARDING problems are FPT with respect to the number of guards that are required to guard the terrain when T is an orthogonal terrain.

This discussion leads us to the two major open problems in this topic.

- Does there exists a FPT algorithm whose parameter is the number of guards that are required to guard the orthogonal terrain (say k), which is polynomial over k?
- Are the DISCRETE TERRAIN GUARDING and CONTINUOUS TERRAIN GUARDING problems FPT with respect to the same parameter even when the terrain is not orthogonal?

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