Annihilators of Banach Spaces

K Prahlad Narasimhan

January, 2021

In this report, we will define an annihilator of a subspace of a Banach space and look at the properties of annihilators of closed subspaces of a Banach space. Finally, we will look to characterize the conditions required for the sum of two closed subspaces of a Banach space to be closed. Throughout this report, we let $\mathcal{B}$ denote a Banach space over $\mathbb{R}$ or $\mathbb{C}$. We let $\mathcal{B}^*$ denote the dual of a the Banach space $\mathcal{B}$. We also denote the open and closed balls of radius r around a point $v \in \mathcal{B}$ by $B_r^{\mathcal{B}}(v)$ and $\overline{B_r^{\mathcal{B}}}(v)$ respectively (the metric defined on $\mathcal{B}$ is the one induced by the norm describing $\mathcal{B}$).

Definition. Consider a subspace W of $\mathcal{B}$. The *annihilator* of W, denoted by $W^\perp$, is the set

$$W^\perp = \{f \in \mathcal{B}^* \mid f(v) = 0 \text{ for all } v \in W\}$$

Similarly, for a subspace Z of $\mathcal{B}^*$,

$$Z^\perp = \{v \in \mathcal{B} \mid f(v) = 0 \text{ for all } f \in Z\}$$

Lemma 1. Let W be a subspace of $\mathcal{B}$ and $f \in \mathcal{B}^*$. Then, $d(f, W^\perp) = \sup_{w \in W, ||x|| \leq 1} |f(w)|$.

Proof. Let $g \in W^\perp$. Then,

$$\begin{aligned} ||f - g|| &= \sup_{||w|| \leq 1} |f(w) - g(w)| \\ &\geq \sup_{w \in W, ||w|| \leq 1} |f(w) - g(w)| \\ &\geq \sup_{w \in W, ||w|| \leq 1} |f(w)| && (g(w) = 0) \end{aligned}$$

Thus, $d(f, W^\perp) \geq \sup_{w \in W, ||x|| \leq 1} |f(w)|$. We now prove that there exists a $\psi \in W^\perp$ such that the equality is attained. By the *Hahn-Banach Theorem*[1], there exists a $g \in \mathcal{B}^*$ such that ψ agrees with f in W and $||f|_W||_{W^*} = ||\psi||$. Then, since $f - \psi \in W^\perp$,

$$d(f, W^\perp) \leq ||f - (f - \psi)|| = ||\psi|| = ||f|_W||_{W^*} = \sup_{w \in W, ||w|| \leq 1} |f(w)|$$

[1]Theorem 3.1.2 in "*Functional Analysis*" by S. Kesavan.

This completes the proof of the lemma. $\square$

Lemma 2. Let W be a subspace of $\mathcal{B}$ and Z be a subspace of $\mathcal{B}^*$. Then, $W^\perp$ and $Z^\perp$ are closed subspaces of $\mathcal{B}^*$ and $\mathcal{B}$ respectively.

Proof. First, we prove that $W^\perp$ is a closed subspace of $\mathcal{B}^*$. Let $f, g \in W^\perp$ and $\alpha \in \mathbb{R}$. Clearly, for all $v \in W$,

$$(f + \alpha g)(v) = f(v) + \alpha \cdot g(v) = 0$$

Thus, $W^\perp$ is a subspace of $\mathcal{B}^*$. Now, let $\{f_n\}_{n\in\mathbb{N}} \subseteq W^\perp$ such that $f_n \to f$ for some $f \in \mathcal{B}^*$. Then, since $f_n(v) = 0$ for all $v \in W$, $f(v) = 0$ for all such v. This implies that $f \in W^\perp$ and hence proves that $W^\perp$ is a closed subspace of $\mathcal{B}^*$.

Now, for $u, v \in Z^\perp$, and $\alpha \in \mathbb{R}$, for all $f \in Z$,

$$f(u + \alpha v) = f(u) + \alpha \cdot f(v) = 0$$

Thus, $Z^\perp$ is a subspace of $\mathcal{B}$. For a sequence $\{v_n\}$ in $Z^\perp$ converging to some $v \in \mathcal{B}$, for all $f \in Z$, $f(v) = 0$ since each such f is continuous. Thus, $v \in Z^\perp$. This implies that $Z^\perp$ is closed. $\square$

Corollary 3. Let W be a subspace of $\mathcal{B}$ and Z be a subspace of $\mathcal{B}^*$. Then, $\overline{W} \subseteq (W^\perp)^\perp$ and $\overline{Z} \subseteq (Z^\perp)^\perp$.

Proof. Let $v \in W$. Then, for all $f \in W^\perp$, $f(v) = 0$. Thus, $W \subseteq (W^\perp)^\perp$. Since $(W^\perp)^\perp$ is closed by Lemma 2, $\overline{W} \subseteq (W^\perp)^\perp$. Similarly, for all $f \in Z$, $f(v) = 0$ for all $v \in Z^\perp$. Thus, $Z \subseteq (Z^\perp)^\perp$ implying that $\overline{Z} \subseteq (Z^\perp)^\perp$. $\square$

Corollary 4. Let W be a subspace of $\mathcal{B}$. Then, $\overline{W} = (W^\perp)^\perp$.

Proof. By Corollary 3, we know that $\overline{W} \subseteq (W^\perp)^\perp$. We know, for $v \notin \overline{W}$, there exists a $f \in \mathcal{B}^*$ such that $f(v) > 0$ and $f(w) = 0$ for all $w \in W$[2]. This proves that there exists a $f \in W^\perp$ such that $f(v) \neq 0$. Thus, $v \notin (W^\perp)^\perp$. This proves that $(W^\perp)^\perp \subseteq W$ which completes the proof of our claim. $\square$

Observation 5. For subspaces W_1, W_2 of $\mathcal{B}$ such that $W_1 \subseteq W_2$, ${W_2}^\perp \subseteq {W_1}^\perp$. Similarly, for subspaces Z_1, Z_2 of $\mathcal{B}^*$ such that $Z_1 \subseteq Z_2$, ${Z_2}^\perp \subseteq {Z_1}^\perp$.

Lemma 6. Let W_1 and W_2 be subspaces of $\mathcal{B}$. Then, $W_1 \cap W_2 \subseteq ({W_1}^\perp + {W_2}^\perp)^\perp$. Similarly, for subspaces Z_1 and Z_2 of $\mathcal{B}^*$, $Z_1 \cap Z_2 \subseteq ({Z_1}^\perp + {Z_2}^\perp)^\perp$.

Proof. Let $w \in W_1 \cap W_2$. Then, for all $f_1 + f_2 \in {W_1}^\perp + {W_2}^\perp$ where $f_1 \in {W_1}^\perp$ and $f_2 \in {W_2}^\perp$,

$$(f_1 + f_2)(w) = f_1(v) + f_2(w) = 0$$

Thus, $W_1 \cap W_2 \subseteq ({W_1}^\perp + {W_2}^\perp)^\perp$.

[2] Corollary 3.2.1 in "*Functional Analysis*" by S. Kesavan.

Let $v \in {Z_1}^\perp + {Z_2}^\perp$. Then, $v \in {Z_1}^\perp$ and $v \in {Z_2}^\perp$ since 0 belongs to the annihilator of any subspace of $\mathcal{B}^*$. Thus, for all $f \in Z_1 \cap Z_2$, $f(v) = 0$. Hence, $Z_1 \cap Z_2 \subseteq ({Z_1}^\perp + {Z_2}^\perp)^\perp$. □

Corollary 7. Let W_1 and W_2 be closed subspaces of $\mathcal{B}$. Then,

$$W_1 \cap W_2 = ({W_1}^\perp + {W_2}^\perp)^\perp \text{ and } {W_1}^\perp \cap {W_2}^\perp = (W_1 + W_2)^\perp$$

Proof. By Lemma 6, $W_1 \cap W_2 \subseteq ({W_1}^\perp + {W_2}^\perp)^\perp$. Since 0 belongs to the annihilator of any subspace of $\mathcal{B}$, ${W_1}^\perp, {W_2}^\perp \subseteq {W_1}^\perp + {W_2}^\perp$. Thus, by Observation 5 and Corollary 4, $({W_1}^\perp + {W_2}^\perp)^\perp \subseteq ({W_1}^\perp)^\perp = \overline{W_1} = W_1$. Similarly, $({W_1}^\perp + {W_2}^\perp)^\perp \subseteq W_2$. This proves that $({W_1}^\perp + {W_2}^\perp)^\perp \subseteq W_1 \cap W_2$ and hence proves the first part of our claim.

By Lemma 6 and Corollary 4,

$${W_1}^\perp \cap {W_2}^\perp \subseteq (({W_1}^\perp)^\perp + ({W_2}^\perp)^\perp)^\perp = (\overline{W_1} + \overline{W_2})^\perp = (W_1 + W_2)^\perp$$

Since $W_1 + W_2$ is a superset of W_1 and W_2, by Observation 5, we have that ${W_1}^\perp \cap {W_2}^\perp \supseteq (W_1 + W_2)^\perp$. This completes the proof of our claim. □

Corollary 8. Let W_1 and W_2 be closed subspaces of $\mathcal{B}$. Then,

$$(W_1 \cap W_2)^\perp \supseteq \overline{({W_1}^\perp + {W_2}^\perp)} \text{ and } ({W_1}^\perp \cap {W_2}^\perp)^\perp = \overline{(W_1 + W_2)}$$

Proof. The first part of the claim follows directly from Corollary 7 and Corollary 3 while the second part of the claim follows from Corollary 7 and Corollary 4. □

Observation 9. Let W_1 and W_2 be two closed subspaces of $\mathcal{B}$. Then, $W_1 \times W_2$ with following norm is a Banach space.

$$||(w_1, w_2)||_{W_1 \times W_2} = ||w_1|| + ||w_2||$$

Corollary 10. Let W_1 and W_2 be two closed subspaces of $\mathcal{B}$. Let $f \colon W_1 \times W_2 \mapsto W_1 + W_2$ such that $f(w_1, w_2) = w_1 + w_2$ where $W_1 \times W_2$ is endowed with the topology induced by the norm described in Observation 9. Then, f is an open map.

Proof. Clearly,

$$||f(w_1, w_2)|| = ||w_1 + w_2|| \leq ||w_1|| + ||w_2|| = ||(w_1, w_2)||_{W_1 \times W_2}$$

Thus, f is a continuous linear transformation. Also, f is surjective. Thus, by the *Open Mapping Theorem*[3], f is open. □

[3]Theorem 4.4.1 in "*Functional Analysis*" by S. Kesavan.

Lemma 11. Let W_1 and W_2 be two closed subspaces of $\mathcal{B}$ such that $W_1 + W_2$ is also closed. Then, there exists a $\alpha > 0$ such that for all $z = w_1 + w_2 \in W_1 + W_2$, where $w_1 \in W_1$ and $w_2 \in W_2$,

$$||w_1|| \leq \alpha \cdot ||z|| \text{ and } ||w_2|| \leq \alpha \cdot ||z||$$

Proof. Let $f : W_1 \times W_2 \mapsto W_1 + W_2$ such that $f(w_1, w_2) = w_1 + w_2$. Then, by Corollary 10, f is open. Then, $f(B_1^{W_1 \times W_2}(0))$ is open in $W_1 + W_2$. Hence, there exists a $c > 0$ such that $B_c^{W_1+W_2}(0) \subseteq f(B_1^{W_1 \times W_2}(0))$. That is, there exists a $c > 0$ such that for all $w \in W_1 + W_2$ with $||w|| < c$, w can be written as $w_1 + w_2$ for some $w_1 \in W_1$ and $w_2 \in W_2$, where $||(w_1, w_2)||_{W_1 \times W_2} = ||w_1|| + ||w_2|| < 1$. Now, for any non-zero $w \in W_1 + W_2$, consider the vector $\frac{c}{2||w||} w$. Then, $\frac{c}{2||w||} w = w_1 + w_2$ where $w_1 \in W_1$ and $w_2 \in W_2$ with $||w_1|| < 1$ and $||w_2|| < 1$. Thus, if $\alpha = \frac{2}{c}$ and $w_1' = \alpha w_1 \in W_1$ while $w_2' = \alpha w_2 \in W_2$; then, $w = w_1' + w_2'$ and $||w_1'||$ and $||w_2'||$ are bounded above by $\alpha \cdot ||w||$. This completes the proof of this lemma. □

Theorem 12. Let W_1 and W_2 be two closed subspaces of $\mathcal{B}$ such that $W_1 + W_2$ is also closed. Then, there exists a $\alpha > 0$ such that $d(v, W_1 \cap W_2) \leq \alpha \cdot (d(v, W_1) + d(v, W_2))$ for all $v \in \mathcal{B}$.

Proof. Let $v \in \mathcal{B}$ and $\epsilon > 0$. By definition, we have $w_1 \in W_1$ and $w_2 \in W_2$ such that

$$d(v, w_1) \leq d(v, W_1) + \epsilon \text{ and } d(v, w_2) \leq d(v, W_2) + \epsilon \tag{1}$$

Let $z = w_1 - w_2 \in W_1 + W_2$. Then, by Lemma 11, there exists a $c > 0$, $w_1' \in W_1$, $w_2' \in W_2$ such that

$$w_1 - w_2 = w_1' + w_2' \tag{2}$$
$$||w_1'|| \leq c \cdot ||w_1 - w_2|| \tag{3}$$
$$||w_2'|| \leq c \cdot ||w_1 - w_2|| \tag{4}$$

By Equation (2), $w_1 - w_1' = w_2 - w_2'$. Hence, $w_1 - w_1' = w_2 - w_2' \in W_1 \cap W_2$. Thus,

$$\begin{aligned}
d(v, W_1 \cap W_2) &\leq d(v, w_1 - w_1') \\
&\leq ||v - w_1 + w_1'|| \\
&\leq ||v - w_1|| + ||w_1'|| \\
&\leq ||v - w_1|| + c \cdot ||w_1 - w_2|| && \text{(By Equation (3))} \\
&= ||v - w_1|| + c \cdot ||(w_1 - v) + (v - w_2)|| \\
&\leq (1 + c) \cdot ||v - w_1|| + c \cdot ||(v - w_2)|| && \text{(By Equation (4))} \\
&\leq (1 + c) \cdot d(v, w_1) + (1 + c) \cdot d(v, w_2) \\
&\leq (1 + c) \cdot (d(v, W_1) + \cdot d(v, W_2)) + 2(1 + c) \cdot \epsilon && \text{(By Equation (1))}
\end{aligned}$$

Since ϵ can be arbitrarily small, letting α be $1 + c$ completes the proof. □

Corollary 13. Let Z_1 and Z_2 be two closed subspaces of $\mathcal{B}^*$ such that $Z_1 + Z_2$ is also closed. Then, there exists a $\alpha > 0$ such that $d(f, Z_1 \cap Z_2) \leq \alpha \cdot (d(f, Z_1) + d(f, Z_2))$ for all $f \in \mathcal{B}^*$.

Proof. Since Lemma 11 and Theorem 12 can be stated in terms of $\mathcal{B}^*$, our claim follows. □

Corollary 14. Let W_1 and W_2 be two closed subspaces of $\mathcal{B}$ such that ${W_1}^{\perp} + {W_2}^{\perp}$ is closed. Then, there exists a $\alpha > 0$ such that for all $f \in \mathcal{B}^*$,

$$\sup_{w \in \overline{W_1 + W_2}, ||w|| \leq 1} |f(w)| \leq \alpha \cdot (\sup_{w_1 \in W_1, ||w_1|| \leq 1} |f(w_1)| + \sup_{w_2 \in W_2, ||w_2|| \leq 1} |f(w_2)|)$$

Proof. Since $(W_1 + W_2)^{\perp} = {W_1}^{\perp} \cap {W_2}^{\perp}$ by Corollary 7 and distance between a point and any set is the same as the distance between that point and the set's closure in any metric space, this claim follows directly from Corollary 13 and Lemma 1. □

Lemma 15. Let W_1 and W_2 be two closed subspaces of a real Banach space $\mathcal{B}$ such that ${W_1}^{\perp} + {W_2}^{\perp}$ is closed. Then, there exists a $\alpha > 0$ such that

$$\frac{1}{\alpha} B_1^{\overline{W_1 + W_2}}(0) \subseteq \overline{B_1^{W_1}(0) + B_1^{W_2}(0)}$$

Proof. Let α be a positive real such that for all $f \in \mathcal{B}^*$,

$$\sup_{w \in \overline{W_1 + W_2}, ||w|| \leq 1} |f(w)| \leq \alpha \cdot (\sup_{w_1 \in W_1, ||w_1|| \leq 1} |f(w_1)| + \sup_{w_2 \in W_2, ||w_2|| \leq 1} |f(w_2)|)$$

Such a α exists by Corollary 14. Assume that there is a $w_0 \in \frac{1}{\alpha} B_1^{\overline{W_1 + W_2}}(0)$ which is not in $\overline{B_1^{W_1}(0) + B_1^{W_2}(0)}$. Then, $w_0 \in \overline{W_1 + W_2}$ and $||w_0|| < \frac{1}{\alpha}$. Since sum of two convex sets is convex and a closure of a convex set is convex[4], $\overline{B_1^{W_1}(0) + B_1^{W_2}(0)}$ is a convex set. Thus, there exists a hyperplane that strictly separates w_0 and $B_1^{W_1}(0) + B_1^{W_2}(0)$[5]. That is, there exists a $f \in \mathcal{B}^*$ and $a \in \mathbb{R}$ such that $f(w_1 + w_2) < a < f(w_0)$ for all $w_1 \in B_1^{W_1}(0)$ and $w_2 \in B_1^{W_2}(0)$. Since $0 \in B_1^{W_1}(0) + B_1^{W_2}(0)$, $0 < a < f(w_0)$. Since $w_1 \in B_1^{W_1}(0)$ and $w_2 \in B_1^{W_2}(0)$ implies that the $-w_1$ and $-w_2$ belong to the respective balls, for all such w_1 and w_2, $|f(w_1 + w_2)| > a$. Thus,

$$\begin{aligned}
\sup_{w_1 + w_2 \in B_1^{W_1}(0) + B_1^{W_2}(0)} f(w_1 + w_2) &= \sup_{w_1 \in B_1^{W_1}(0)} f(w_1) + \sup_{w_2 \in B_1^{W_2}(0)} f(w_2) \\
&= \sup_{w_1 \in B_1^{W_1}(0)} |f(w_1)| + \sup_{w_2 \in B_1^{W_2}(0)} |f(w_2)| \\
&= \sup_{w_1 \in W_1, ||w_1|| \leq 1} |f(w_1)| + \sup_{w_2 \in W_2, ||w_2|| \leq 1} |f(w_2)|
\end{aligned}$$

Since $\sup_{w_1 + w_2 \in B_1^{W_1}(0) + B_1^{W_2}(0)} f(w_1 + w_2) \leq a < f(w_0)$, it follows that

$$\sup_{w_1 \in W_1, ||w_1|| \leq 1} |f(w_1)| + \sup_{w_2 \in W_2, ||w_2|| \leq 1} |f(w_2)| < f(w_0) = ||w_0|| f(\frac{w_0}{||w_0||}) < \frac{1}{\alpha} \cdot \sup_{w \in \overline{W_1 + W_2}, ||w|| \leq 1} f(w)$$

This contradicts our premise and thus completes the proof of this lemma. □

[4]Proposition 1.1.1 in "*Convex Optimization Theory*" by Bertsekas D. P.
[5]Theorem 3.2.2 in "*Functional Analysis*" by S. Kesavan.

Corollary 16. Let W_1 and W_2 be two closed subspaces of a complex Banach space $\mathcal{B}$ such that ${W_1}^{\perp} + {W_2}^{\perp}$ is closed. Then, there exists a $\alpha > 0$ such that

$$\frac{1}{\alpha} B_1^{\overline{W_1+W_2}}(0) \subseteq \overline{B_1^{W_1}(0) + B_1^{W_2}(0)}$$

Proof. The proof of this result follows from using the proof of Lemma 15 on the real part of a complex function $f \in \mathcal{B}^*$ and noting that $||f|| = ||\mathrm{Re}(f)||$[6]. □

Observation 17. Let W_1 and W_2 be two closed subspaces of a Banach space $\mathcal{B}$ such that ${W_1}^{\perp} + {W_2}^{\perp}$ is closed. Then, there exists a $\alpha > 0$ such that

$$\frac{1}{\alpha} B_1^{\overline{W_1+W_2}}(0) \subseteq \overline{B_1^{W_1}(0) + B_1^{W_2}(0)}$$

Theorem 18. Let W_1 and W_2 be two closed subspaces of $\mathcal{B}$. Then, the following are equivalent:

(i) $W_1 + W_2$ is closed.

(ii) ${W_1}^{\perp} + {W_2}^{\perp}$ is closed.

(iii) $W_1 + W_2 = \left({W_1}^{\perp} \cap {W_2}^{\perp}\right)^{\perp}$.

(iv) ${W_1}^{\perp} + {W_2}^{\perp} = (W_1 \cap W_2)^{\perp}$.

Proof. By Corollary 8, it is clear that (i) and (iii) are equivalent. Also, (iv) $\implies$ (ii) follows directly from Lemma 2. We complete the proof of our claim by showing that (i) $\implies$ (iv) and (ii) $\implies$ (i).

Claim 18.1. If W_1 and W_2 be two closed subspaces of $\mathcal{B}$ such $W_1 + W_2$ is closed, ${W_1}^{\perp} + {W_2}^{\perp} = (W_1 \cap W_2)^{\perp}$.

Proof of Claim. From Corollary 8, we know that

$${W_1}^{\perp} + {W_2}^{\perp} \subseteq \overline{({W_1}^{\perp} + {W_2}^{\perp})} \subseteq (W_1 \cap W_2)^{\perp}$$

So, we are left to prove that $(W_1 \cap W_2)^{\perp} \subseteq {W_1}^{\perp} + {W_2}^{\perp}$. Let $f \in (W_1 \cap W_2)^{\perp}$. Define $\psi \colon W_1 + W_2 \mapsto \mathbb{K}$, where $\mathbb{K}$ is the underlying field of $\mathcal{B}$, by $\psi(w_1 + w_2) = f(w_1)$. First, we will prove that ψ is a well defined map. Then, we show that ψ is a continuous linear functional on $W_1 + W_2$.

If $w_1 + w_2 = w_1' + w_2'$ for some other $w_1' \in W_1$ and $w_2' \in W_2$, then, $w_1' - w_1 = w_2' - w_2 \in W_1 \cap W_2$. Thus, $f(w_1 - w_1') = f(w_1) - f(w_1') = 0$. This implies that ψ is a well defined function. Clearly, ψ is linear. Since $W_1 + W_2$ is closed, by Lemma 11, there is a $\alpha > 0$ such that

$$\begin{aligned} ||w_1|| \leq \alpha \cdot ||w_1 + w_2|| &\implies ||f|| \cdot ||w_1|| \leq \alpha \cdot ||f|| \cdot ||w_1 + w_2|| \\ &\implies ||f(w_1)|| \leq \alpha \cdot ||f|| \cdot ||w_1 + w_2|| \\ &\implies ||\psi(w_1 + w_2)|| \leq k \cdot ||w_1 + w_2|| \qquad \text{(Let } k = \alpha \cdot ||f||\text{)} \end{aligned}$$

[6]Proposition 3.1.1 in "*Functional Analysis*" by S. Kesavan.

This implies that ψ is continuous. Since $0 \in W_2$, ψ agrees with f in W_1. Since $0 \in W_1$, $\psi(w_2) = 0$ for all $w_2 \in W_2$. By the *Hahn-Banach Theorem*[7], there exists a continuous extension of ψ to $\mathcal{B}^*$, say $\psi_0 \in \mathcal{B}^*$. Since ψ_0 agrees with ψ in W_1, it agrees with f in W_1. Thus, $f - \psi_0 \in {W_1}^\perp$. Since ψ_0 agrees with ψ in W_2, it vanishes in W_2. Hence, $\psi_0 \in {W_2}^\perp$. This implies that $f = (f - \psi_0) + \psi_0 \in {W_1}^\perp + {W_2}^\perp$ proving that $(W_1 \cap W_2)^\perp \subseteq {W_1}^\perp + {W_2}^\perp$. $\Diamond$

This proves that (i) $\implies$ (iv). We now prove that (ii) $\implies$ (i).

Claim 18.2. If W_1 and W_2 be two closed subspaces of $\mathcal{B}$ such ${W_1}^\perp + {W_2}^\perp$ is closed, $W_1 + W_2$ is closed.

Proof of Claim. Endow $W_1 \times W_2$ with the norm $||\cdot||_{W_1 \times W_2}$ where $||(w_1, w_2)||_{W_1 \times W_2} = \max\{||w_1||, ||w_2||\}$. Define f from $W_1 \times W_2$ to $\overline{W_1 + W_2}$ by $f(w_1, w_2) = w_1 + w_2$. Clearly, f is linear. For all $(w_1, w_2) \in W_1 \times W_2$,

$$||w_1 + w_2|| \leq ||w_1|| + ||w_2|| \leq 2 \cdot \max\{||w_1||, ||w_2||\}$$

Thus, f is continuous. Note that $B_1^{W_1 \times W_2}(0) = \{(w_1, w_2) \mid ||w_1|| < 1, ||w_2|| < 1\}$. Hence, we have,

$$f(B_1^{W_1 \times W_2}(0)) = \{w_1 + w_2 \mid ||w_1|| < 1, ||w_2|| < 1\} = B_1^{W_1}(0) + B_1^{W_2}(0)$$

By Observation 17, there exists a $\alpha > 0$ such that

$$\begin{aligned} \frac{1}{\alpha} B_1^{\overline{W_1+W_2}}(0) \subseteq \overline{f(B_1^{W_1 \times W_2}(0))} &\implies \frac{1}{2\alpha} B_1^{\overline{W_1+W_2}}(0) \subseteq f(B_1^{W_1 \times W_2}(0)) \\ &\implies B_{\frac{1}{2\alpha}}^{\overline{W_1+W_2}}(0) \subseteq f(B_1^{W_1 \times W_2}(0)) \\ &\implies f \text{ is open.} \end{aligned}$$

Since f is an open linear map, it is surjective. This gives

$$\overline{W_1 + W_2} = f(W_1 \times W_2) \subseteq W_1 + W_2 \subseteq \overline{W_1 + W_2} \implies W_1 + W_2 = \overline{W_1 + W_2}$$

Thus, $W_1 + W_2$ is closed. $\Diamond$

This completes the proof of this theorem. $\Box$

[7]Theorem 3.1.2 in "*Functional Analysis*" by S. Kesavan